On Fields of Definition of Components of the Siegel Supersingular Locus

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ABSTRACT. Recently Ibukiyama proves an explicit formula for the number of certain non-principal polarizations on a superspecial abelian surface, extending his earlier work with Katsura for principal polarizations [Compos. Math. 1994]. As a consequence of Ibukiyama's formula, there exists a geometrically irreducible component of the Siegel supersingular locus which is defined over the prime finite field. In this note we give a direct proof of this result.

1. INTRODUCTION

Let p be a rational prime number, and let $n \geq 1$ be a positive integer. Let $\mathcal{A}_n$ denote the coarse moduli space over $\mathbb{F}_p$ of n-dimensional principally polarized abelian varieties. The supersingular locus of $\mathcal{A}_n \otimes \overline{\mathbb{F}}_p$ is denoted by $\overline{\mathcal{S}}_n$, which is the closed reduced $\overline{\mathbb{F}}_p$-subscheme consisting of all supersingular points in $\mathcal{A}_n(\overline{\mathbb{F}}_p)$. An abelian variety over $\overline{\mathbb{F}}_p$ is *supersingular* (resp. *superspecial*) if it is isogenous (resp. isomorphic) to a product of supersingular elliptic curves. It is known that $\overline{\mathcal{S}}_n$ is defined over $\mathbb{F}_p$, i.e. the action of the Galois group $\Gamma := \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ leaves the set $\overline{\mathcal{S}}_n$ stable. The unique model of $\overline{\mathcal{S}}_n$ over $\mathbb{F}_p$ in $\mathcal{A}_n$ is denoted by $\mathcal{S}_n$, which is called the supersingular locus of $\mathcal{A}_n$. The set of irreducible components of $\overline{\mathcal{S}}_n$ is denoted by $\Pi_0(\overline{\mathcal{S}}_n)$, on which Γ operates. An irreducible component $V \in \Pi_0(\overline{\mathcal{S}}_n)$ is defined over $\mathbb{F}_p$ if and only if V is stable under the action of Γ. In this note we give a proof of the following result.

Theorem 1.1 (Li-Oort, Ibukiyama)**.** *There exists an irreducible component of $\overline{\mathcal{S}}_n$ that is defined over $\mathbb{F}_p$.*

In Section 2, we shall give some background knowledge on $\Pi_0(\overline{\mathcal{S}}_n)$ due to Li and Oort [5]. Based on loc. cit., Theorem 1.1 is reduced to non-emptiness of the set of certain polarized superspecial abelian varieties that admit a model defined over $\mathbb{F}_p$ (the set $\Lambda_n(\mathbb{F}_p)$ or $\Sigma_n(\mathbb{F}_p)$ in Subsection 2.1); see Theorem 2.3. It follows that Theorem 1.1 is trivial when n is odd. When n is even, Theorem 1.1 then follows from the following result.

Theorem 1.2. *There exists a polarized superspecial abelian surface (A, λ) over $\mathbb{F}_p$ such that $\ker \lambda = A[F]$, where $F : A \to A^{(p)} = A$ is the relative Frobenius morphism on A.*

In [1, 2] Ibukiyama gives an explicit formula for the cardinality of $\Sigma_2(\mathbb{F}_p)$, and as a byproduct he obtains Theorem 1.2. The result of Ibukiyama confirms the

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existence of polarized abelian surfaces as in Theorem 1.2. We construct such an polarized abelian surface directly in Section 3. This proves Theorem 1.1 by a different method.

2. Preliminaries and background

2.1. **Finite sets Λ_n and Σ_n.** Let $\Gamma := \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ denote the Galois group over $\mathbb{F}_p$, and put $\Gamma_2 := \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$. The Frobenius automorphism $a \mapsto a^p$ in Γ is denoted by σ_p. For any positive integer $n \geq 1$, let Λ_n denote the set of isomorphism classes of superspecial principally polarized abelian varieties of dimension n over $\overline{\mathbb{F}}_p$. This is a finite set on which Γ acts. It is known that the subgroup Γ_2 acts trivially on Λ_n so the action factors through the quotient $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$. Put

$$\Lambda_n(\mathbb{F}_p) := \Lambda_n^\Gamma \subset \Lambda_n,$$

the subset of elements fixed by σ_p. Let E_0 be a supersingular elliptic curve over $\mathbb{F}_p$ such that the endomorphism endomorphism π_{E_0} of E_0 satisfies $\pi_{E_0}^2 = -p$. Put $A_0 = E_0^n$ and take λ_0 to be the product of copies of the canonical principal polarization of E_0. Clearly, (A_0, λ_0) is a superspecial principally polarized abelian variety of dimension n over $\mathbb{F}_p$. In particular, the set $\Lambda_n(\mathbb{F}_p)$ is nonempty.

For any even positive integer $n = 2c$, denote by Σ_n the set of isomorphism classes of superspecial polarized abelian varieties (A, λ) of dimension n over $\overline{\mathbb{F}}_p$ such that $\ker \lambda = A[F]$, where $A^{(p)}$ is the base change of $\operatorname{Spec} \overline{\mathbb{F}}_p \to \operatorname{Spec} \overline{\mathbb{F}}_p$ induced by σ_p, and $F : A \to A^{(p)}$ is the relative Frobenius morphism. Clearly, for any $(A, \lambda) \in \Sigma_n$, one has $\deg \lambda = p^n$. The Galois group Γ acts on Σ_n which factors through $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$. Indeed, any superspecial abelian variety over $\overline{\mathbb{F}}_p$ admits a model A_0' over $\mathbb{F}_{p^2}$ such that any endomorphism of A is defined over $\mathbb{F}_{p^2}$ (for the $\mathbb{F}_{p^2}$-structure of A_0'), in particular, any polarization of A is defined over $\mathbb{F}_{p^2}$. Thus, the group Γ_2 acts trivially on Σ_n. Similarly, we define

$$\Sigma_n(\mathbb{F}_p) := \Sigma_n^\Gamma \subset \Sigma_n,$$

the subset of elements fixed by σ_p. Unlike Λ_n, it is not clear whether $\Sigma_n(\mathbb{F}_p)$ is nonempty a priori.

Recall that the field of moduli of a polarized abelian variety (A, λ) over $\overline{\mathbb{F}}_p$ is the field of definition of the isomorphism class $[(A, \lambda)]$, that is, the smallest finite field $\mathbb{F}_{p^a}$ such that $(A, \lambda) \simeq ({}^\sigma A, {}^\sigma \lambda)$ over $\overline{\mathbb{F}}_p$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^a})$.

Lemma 2.1. *Let (A, λ) be any polarized abelian variety over $\overline{\mathbb{F}}_p$, and suppose that the field of moduli of (A, λ) is $\mathbb{F}_{p^a}$. Then (A, λ) has a model defined over $\mathbb{F}_{p^a}$.*

Proof. The lemma is proved for A being superspecial [2, Lemma 4.1]. The same proof works for an arbitrary abelian variety; also see [7, Prop. 6.3] for a similar proof. ∎

Lemma 2.2. *If an element (A, λ) in Σ_n has a model (A', λ') defined over $\mathbb{F}_p$, then* $\ker \lambda' = A'[F]$.

Proof. We use the following basic fact: if $H_1 < H_2$ are two finite group schemes over a field and if $\operatorname{rank} H_1 = \operatorname{rank} H_2$, then $H_1 = H_2$. It follows that if $\varphi : A \to B$ is an isogeny of abelian varieties over k, K/k is a field extension and $\varphi_K : A_K \to$

B_K is the base change morphism, then $\ker \varphi_K = \ker \varphi \otimes_k K$. Applying this for $F : A \to A^{(p)}$ and $K/k = \overline{\mathbb{F}}_p/\mathbb{F}_p$, we get

$$\ker \lambda' \otimes \overline{\mathbb{F}}_p = \ker \lambda = A[F] = A'[F] \otimes \overline{\mathbb{F}}_p.$$

By the faithfully flat descent, we have $\ker \lambda' = A'[F]$. ∎

Let $\widetilde{\Sigma}_n(\mathbb{F}_p)$ be the set of isomorphism classes of superspecial polarized abelian varieties (A, λ) of dimension n (n being even) over $\mathbb{F}_p$ such that $\ker(\lambda) = A[F]$. It follows from Lemmas 2.1 and 2.2 that the natural map $\widetilde{\Sigma}_n(\mathbb{F}_p) \to \Sigma_n(\mathbb{F}_p)$ is surjective. Particularly, the set $\Sigma_n(\mathbb{F}_p)$ is nonempty if and only if so is $\widetilde{\Sigma}_n(\mathbb{F}_p)$.

2.2. **Fields of definition for components.** As in Section 1, we let $\mathcal{A}_n$ denote the coarse moduli space over $\mathbb{F}_p$ of n-dimensional principally polarized abelian varieties, and let $\mathcal{S}_n \subset \mathcal{A}_g$ be the supersingular locus of of $\mathcal{A}_g$. Li-Oort proved [5, 4.9, p. 26] that there is a one-to-one correspondence between the set $\Pi_0(\overline{\mathcal{S}}_n)$ of irreducible components of $\overline{\mathcal{S}}_n := \mathcal{S}_n \otimes \overline{\mathbb{F}}_p$ and either the set Λ_n or the set Σ_n according as n is odd or even. The Galois group Γ operates on $\Pi_0(\overline{\mathcal{S}}_n)$ as well as on the sets Λ_n and Σ_n. We claim that the correspondence in loc. cit. respects the action of Γ. Let $V \in \Pi_0(\overline{\mathcal{S}}_n)$ be an irreducible component. It is proved in [5, 4.9 (iii), p. 26] that the subset U of supergeneral abelian varieties in the sense of Li and Oort (i.e. with $a = 1$) is an open and dense subset in V. For any point (B, λ_B) in U, there is a unique "PFTQ" (polarized flag type quotient) up to equivalence [5, Chapter 3]

$$(A_{n-1}, \eta_{n-1}) \to (A_{n-2}, \eta_{n-2}) \to \cdots \to (A_0, \eta_0)$$

such that $(A_0, \eta_0) \simeq (B, \lambda_B)$ and $(A_{n-1}, \eta_{n-1}) \simeq (A, p^m \lambda)$ for a superspecial polarized abelian variety (A, λ) that lies in Λ_n if n is odd and in Σ_n if n is even. The pair (A, λ) depends only on U up to isomorphism, and corresponds to the component V. It is then clear from this construction that for any $\sigma \in \Gamma$ the conjugate ${}^{\sigma}U$ of U corresponds to the isomorphism class $[({}^{\sigma}A, {}^{\sigma}\lambda)]$. This proves the claim. Note that the isomorphism class $[({}^{\sigma}A, {}^{\sigma}\lambda)]$ is defined over $\mathbb{F}_p$ if and only if $(A.\lambda)$ admits a model defined over $\mathbb{F}_p$ (Lemma 2.1). Thus, we have the following consequence of results of Li and Oort.

Theorem 2.3 (Li-Oort, cf. [2, Theorem 4.4])**.**

(1) *Every irreducible component $V \subset \overline{\mathcal{S}}_n$ is defined over $\mathbb{F}_{p^2}$.*

(2) *Let $\Pi_0(\overline{\mathcal{S}}_n)(\mathbb{F}_p)$ be the set of irreducible components of $\overline{\mathcal{S}}_n$ that are defined over $\mathbb{F}_p$. Then*

$$\Pi_0(\overline{\mathcal{S}}_n)(\mathbb{F}_p) \simeq \begin{cases} \Lambda_n(\mathbb{F}_p) & \text{if } n \text{ is odd;} \\ \Sigma_n(\mathbb{F}_p) & \text{if } n \text{ is even.} \end{cases} \tag{2.1}$$

2.3. **Theorem 1.2 implies Theorem 1.1.** It follows from Theorem 2.3 that Theorem 1.1 is equivalent to that $\Lambda_n(\mathbb{F}_p)$ is nonempty if n is odd and $\Sigma_n(\mathbb{F}_p)$ is nonempty if n is even. Non-emptiness of $\Lambda_n(\mathbb{F}_p)$ is obvious. Thus, it reduces to prove non-emptiness of $\Sigma_n(\mathbb{F}_p)$. But this following from non-emptiness of $\Sigma_2(\mathbb{F}_p)$, which is equivalent to Theorem 1.2 by Lemma 2.1.

2.4. **Explicit formula for $|\Lambda_2(\mathbb{F}_p)|$.** Let χ be the quadratic character associated to the real quadratic field $\mathbb{Q}(\sqrt{p})/\mathbb{Q}$, and let $B_{2,\chi}$ be the second generalized Bernoulli number. For any algebraic number α, we write $h(\alpha)$ for the class number of the number field $\mathbb{Q}(\alpha)$.

Theorem 2.4 (Ibukiyama [1, Theorem 5.2])**.** *One has*

$$|\Sigma_2(\mathbb{F}_p)| = 1, 1, 1, \textit{ for } p = 2, 3, 5, \textit{ respectively.}$$

For $p \geq 7$*, if* $p \equiv 1 \mod 4$*, then*

$$\begin{aligned} |\Sigma_2(\mathbb{F}_p)| =& \frac{1}{2^5 \cdot 3}\left(9 - 2\left(\frac{2}{p}\right)\right) B_{2,\chi} + \frac{1}{2^4} h(\sqrt{-p}) \\ &+ \frac{1}{2^3} h(\sqrt{-2p}) + \frac{1}{2^2 \cdot 3}\left(3 + \left(\frac{2}{p}\right)\right) h(\sqrt{-3p}); \end{aligned} \tag{2.2}$$

if $p \equiv 3 \mod 4$*, then*

$$\begin{aligned} |\Sigma_2(\mathbb{F}_p)| =& \frac{1}{2^5 \cdot 3} B_{2,\chi} + \frac{1}{2^4}\left(1 - \left(\frac{2}{p}\right)\right) h(\sqrt{-p}) \\ &+ \frac{1}{2^3} h(\sqrt{-2p}) + \frac{1}{2^2 \cdot 3} h(\sqrt{-3p}). \end{aligned} \tag{2.3}$$

Non-emptiness of $\Sigma_2(\mathbb{F}_p)$ (or Theorem 1.2) follows immediately from Theorem 2.4, and hence Theorem 1.1 follows.

3. Construction of certain superspecial polarized abelian varieties

In this section we include some results concerning non-emptiness of the set $\widetilde{\Sigma}_n(\mathbb{F}_p)$. The Frobenius endomorphism of an abelian variety A over $\mathbb{F}_q$ will be denoted by π_A. Let $\widetilde{\Sigma}'_n(\mathbb{F}_p)$ be the set of isomorphism classes of superspecial abelian varieties (A, λ) of dimension n over $\mathbb{F}_p$ such that $\pi_A^2 = -p$ and $\ker \lambda = A[F]$. It is clear that non-emptiness of $\widetilde{\Sigma}'_n(\mathbb{F}_p)$ implies that of $\widetilde{\Sigma}_n(\mathbb{F}_p)$.

Lemma 3.1. *Assume that the integer* $n = 2c$ *is divisible by* 4*. Then there is a superspecial polarized abelian variety* (A', λ') *in* $\widetilde{\Sigma}'_n(\mathbb{F}_p)$*.*

Proof. Choose a superspecial abelian variety A_1 over $\mathbb{F}_{p^2}$ of dimension c such that $\pi_{A_1} = -p$. Since every polarization of $A' \otimes \overline{\mathbb{F}}_p$ is defined over $\mathbb{F}_{p^2}$ and the set Σ_c is nonempty, there exists a polarization λ_1 of A_1 such that $\ker \lambda_1 = A_1[F]$. Take

$$(A', \lambda') := \mathrm{Res}_{\mathbb{F}_{p^2}/\mathbb{F}_p}(A_1, \lambda_1). \tag{3.1}$$

Then (A', λ') is a superspecial polarized abelian variety over $\mathbb{F}_p$ of dimension n such that

$$(A', \lambda') \otimes \mathbb{F}_{p^2} = (A_1, \lambda_1) \times (A_1^{(p)}, \lambda_1^{(p)}). \tag{3.2}$$

This gives

$$\ker \lambda' = \ker \lambda_1 \times \ker \lambda_1^{(p)} = A_1[F] \times A_1^{(p)}[F] = A'[F]. \tag{3.3}$$

From (3.2) we get $\pi_{A'}^2 = -p$. ∎

Lemma 3.2. *Assume that* $\left(\frac{-1}{p}\right) = 1$*. Then for any even positive integer* n*, there exists a superspecial polarized abelian variety* (A, λ) *in* $\widetilde{\Sigma}'_n(\mathbb{F}_p)$

Proof. It suffices to prove this for $n = 2$ as we can take the product of copies of such a polarized abelian surface. Let $(A_0, \lambda_0) = (E_0^2, \lambda_0)$ be the superspecial principally polarized abelian surface over $\mathbb{F}_p$ as in Subsection 2.1; one has $\pi_{A_0}^2 = -p$. Consider isogenies $\alpha : (A, \lambda) \to (A_0, \lambda_0)$ of degree p with $\alpha^* \lambda_0 = \lambda$. The family $\{\alpha\}$ forms a projective space $\mathbf{P}^1$ that has an $\mathbb{F}_p$-structure induced from the $\mathbb{F}_p$-structure of A_0. If an isogeny $\alpha : (A, \lambda) \to (A_0, \lambda_0)$ corresponds to a point $[a : b] \in \mathbf{P}^1(\overline{\mathbb{F}}_p)$, then $(A, \lambda) \in \Sigma_2$ if and only if $a^{p+1}+b^{p+1} = 0$ ([4, (3.4), p. 119] and [6, Lemma 4.3]). Since $(-1/p) = 1$, there exists a point $[a : b] \in \mathbf{P}^1(\mathbb{F}_p)$ such that $a^{p+1} + b^{p+1} = a^2 + b^2 = 0$. Then the corresponding isogeny $\alpha : (A, \lambda) \to (A_0, \lambda_0)$ is defined over $\mathbb{F}_p$ and satisfies both $\pi_A^2 = -p$ and $\ker \lambda = A[F]$. ∎

Corollary 3.3. *Assume that* $4|n$ *or* $\left(\frac{-1}{p}\right) = 1$. *Then the set* $\widetilde{\Sigma}'_n(\mathbb{F}_p)$ *is nonempty. Consequently, so is the set* $\widetilde{\Sigma}_n(\mathbb{F}_p)$.

Remark 3.4. We expect that the sufficient conditions in Corollary 3.3 are also necessary for non-emptiness of $\widetilde{\Sigma}'_n(\mathbb{F}_p)$.

For the rest of this section we construct a polarized abelian surface as in Theorem 1.2.

Step 1. There exists a principally polarized abelian surface (A_0, λ_0) over $\mathbb{F}_p$ with $\pi_{A_0}^2 = p$.

Choose an supersingular elliptic curve E over $\mathbb{F}_{p^2}$ such that $\pi_E = p$. Take

$$A_0 = \mathrm{Res}_{\mathbb{F}_{p^2}/\mathbb{F}_p} E.$$

Then the Frobenius endomorphism π_{A_0} of A_0 satisfies $\pi_{A_0}^2 = p$, and one has $A_0 \otimes \mathbb{F}_{p^2} = E \times {}^{\sigma_p}E$. As E admits a principal polarization μ, the product $\lambda_0 := \mu \times {}^{\sigma_p}\mu$ on $A_0 \otimes \mathbb{F}_{p^2}$ is a principal polarization that is defined over $\mathbb{F}_p$.

Step 2. The (covariant) Dieudonné module M_0 of A_0 is a free module over R of rank 2, where $R := \mathbb{Z}_p[\pi_{A_0}] = \mathbb{Z}_p[\sqrt{p}]$. The Frobenius map F and Verschiebung V operate as the multiplication by $\sqrt{p}$. The quasi-polarization induced by λ_0 is a perfect alternating pairing $\langle\, ,\, \rangle : M_0 \times M_0 \to \mathbb{Z}_p$ such that $\langle ax, y\rangle = \langle x, ay\rangle$ for all $a \in R$ and $x, y \in M_0$. We claim that there exists an R-basis e_1, e_2 for M_0 such that

$$\langle e_1, e_2\rangle = \langle \sqrt{p}e_1, \sqrt{p}e_2\rangle = 0 \quad \text{and} \quad \langle e_1, \sqrt{p}e_2\rangle = \langle \sqrt{p}e_1, e_2\rangle = 1. \tag{3.4}$$

That is, $\{e_1, \sqrt{p}e_2, \sqrt{p}e_1, e_2\}$ is a Lagrangian $\mathbb{Z}_p$-basis for the symplectic pairing $\langle\, ,\, \rangle$ on M_0.

Let $K := \mathbb{Q}_p[\sqrt{p}]$ be the fraction field of R. Let

$$\langle\, ,\, \rangle_K : M_0 \times M_0 \to R^\vee, \quad R^\vee := \{x \in K |\, \mathrm{tr}_{K/\mathbb{Q}_p}(xR) \subset \mathbb{Z}_p\} = (2\sqrt{p})^{-1}R.$$

be the unique R-bilinear alternating form such that $\langle x, y\rangle = \mathrm{tr}_{K/\mathbb{Q}}\langle x, y\rangle_K$, where $\mathrm{tr}_{K/\mathbb{Q}_p}$ denotes the reduced trace from K to $\mathbb{Q}_p$. Put $\psi_K(x, y) := (2\sqrt{p})\langle x, y\rangle_K$. Then $\psi_K : M_0 \times M_0 \to R$ is a perfect alternating R-bilinear pairing. Choose an R-basis e_1, e_2 for M_0 such that $\psi_K(e_1, e_2) = 1$. Using the formula $\langle x, y\rangle = \mathrm{tr}_{K/\mathbb{Q}_p}(2\sqrt{p})^{-1}\psi_K(x, y)$, we check (3.4) as follows:

$$\langle \sqrt{p}e_1, e_2\rangle = \langle e_1, \sqrt{p}e_2\rangle = \mathrm{tr}(2\sqrt{p})^{-1}\sqrt{p} = \mathrm{tr}\, 2^{-1} = 1,$$
$$\langle e_1, e_2\rangle = \mathrm{tr}(2\sqrt{p})^{-1} = 0, \quad \langle \sqrt{p}e_1, \sqrt{p}e_2\rangle = \langle e_1, pe_2\rangle = 0.$$

Step 3. Let $\alpha : A \to A_0$ be any isogeny of degree p defined over $\mathbb{F}_p$, and let $\lambda := \alpha^*\lambda_0$. Then A is a superspecial abelian surface over $\mathbb{F}_p$ with $\pi_A^2 = p$, and λ has the property $\ker \lambda = A[F]$.

The Dieudonné module M of A fits into $\mathsf{V}M_0 = \sqrt{p}M_0 \subset M \subset M_0$ with $\dim M/M_0 = 1$. Clearly, M is an R-submodule so that F^2 acts as p on M, and M is superspecial. Put $\overline{M}_0 := M_0/\mathsf{V}M_0 = \mathrm{Span}\{\bar{e}_1, \bar{e}_2\}_{\mathbb{F}_p}$. Define a pairing

$$(\, , \,) : M_0 \times M_0 \to \mathbb{Z}_p, \quad (x, y) := \langle x, \mathsf{F}y \rangle.$$

Clearly, $(M_0, \mathsf{V}M_0) = (\mathsf{V}M_0, M_0) = p\mathbb{Z}_p$. Modulo p, one obtains an $\mathbb{F}_p$-bilinear pairing $(\, , \,) : \overline{M}_0 \times \overline{M}_0 \to \mathbb{F}_p$ satisfying $(\bar{e}_1, \bar{e}_2) = -(\bar{e}_2, \bar{e}_1) = 1$ and $(\bar{e}_1, \bar{e}_1) = (\bar{e}_2, \bar{e}_2) = 0$. Now it is not hard to see that the condition $\ker \lambda = A[F]$ is equivalent to $\mathsf{V}(M^t) = M$, which is also equivalent to the condition $(\overline{M}, \overline{M}) = 0$ (see the proof of [6, Lemma 4.3]). As $\overline{M}$ is generated by a vector $v = a\bar{e}_1 + b\bar{e}_2$, where $a, b \in \mathbb{F}_p$, the condition $(\overline{M}, \overline{M}) = 0$ follows immediately from $(v, v) = 0$. Thus, one proves $\ker \lambda = A[F]$.

This completes the proof of Theorem 1.2

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References

[1] T. Ibukiyama, Quinary lattices and binary quaternion hermitian lattices. Preprint, 2016, 14 pp.
[2] T. Ibukiyama, Type numbers of quaternion hermitian forms and supersingular abelian varieties. Preprint, 2016, 18 pp.
[3] T. Ibukiyama and T. Katsura, On the field of definition of superspecial polarized abelian varieties and type numbers. *Compositio Math.* **91** (1994), 37–46.
[4] T. Katsura and F. Oort, Families of supersingular abelian surfaces, *Compositio Math.* **62** (1987), 107–167.
[5] K.-Z. Li and F. Oort, *Moduli of Supersingular abelian Varieties.* Lecture Notes in Math., vol. 1680, Springer-Verlag, 1998.
[6] C.-F. Yu, The supersingular loci and mass formulas on Siegel modular varieties. *Doc. Math.* **11** (2006), 449–468.
[7] C.-F. Yu, An explicit reciprocity law arising from superspecial abelian varieties and trace formulas. arXiv:1210.1120, 40 pp.

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