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Abstract

In this paper, we study the small-sample statistical condition estimation of the rational Riccati equation, which may be incorporated into a direct solver applying the homotopy method. Our proposed condition estimation algorithms are efficient for small and medium size rational Riccati equations. Numerical examples illustrate the reliability of the algorithms.

Keywords:
Small sample, statistical condition estimation, rational Riccati equations, Fréchet derivative.

1. Introduction

Consider the continuous-time rational Riccati equations (CRREs) arising in the stochastic optimal control of linear time-invariant (LTI) systems [1, 24] with stochastic components. For the numerical solution of CRREs, a Newton-type method is considered in [7, 6]. We propose the homotopy method for CRREs by solving generalized Lyapunov equations and avoid a difficult initial stabilization step [28]. The perturbation analysis and condition number for CRREs are given in [4, 5], but the corresponding computation is impractical for real problems. So we investigate the small-sample statistical condition estimation (SCE) for CRREs, adapting the idea of the SCE in control [18].

The SCE, proposed by Kenny and Laub [19], is a reliable method to estimate the condition numbers. Recently, the SCE has been widely used in many problems, such as linear equations [21], linear least squares problems [2, 20], eigenvalue problems [23], (generalized) Sylvester equations [9, 12], roots of polynomials [22], structured Tikhonov regularization problem [10], the large scale generalized eigenvalues problem [27], generalized spectral projections and matrix sign functions [26], total least squares problem [11] and symmetric algebraic Riccati equations [8].

In this paper, $\mathcal{E}$ denotes the expectation operator and $\mathcal{P}$ the probability. The operator vec stacks the columns of a matrix, with the inverse operator unvec. The 2- and F-norms are denoted respectively by $\|\cdot\|$ and $\|\cdot\|_F$, and I_n is the identity matrix of size $n \times n$.

2. Rational Riccati Equation and Its Perturbation

Consider the control system with state x and control u governed by the Itô differential equation [6, 14, 15]:

$$dx(t) = Ax(t)dt + Bu(t)dt + [A_0x(t) + B_0u(t)]\, d\omega_0(t), \quad x(0) = x_0, \tag{2.1}$$

with the stochastic terms $\{\omega_0(t)\}_{t\in\mathbb{R}_+}$ being independent zero mean real Wiener processes, and the output $y(t) = Cx(t) + Du(t)$. Here $A,\ A_0 \in \mathbb{R}^{n\times n}$, $B,\ B_0 \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$ and $D \in \mathbb{R}^{l\times m}$.

For the stabilization of (2.1), we may choose u to minimize the quadratic cost:

$$J(x_0,u) \equiv \mathcal{E}\int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^\top T \begin{bmatrix} x \\ u \end{bmatrix} dt, \quad T \equiv \begin{bmatrix} Q & L \\ L^\top & R \end{bmatrix} \geq 0,$$

where $T \geq 0$ (i.e., positive semi-definite or psd), as well as $Q = C^\top C \geq 0$ and $R > 0$ (positive definite). This gives rise to the continuous-time rational Riccati equation (CRRE):

$$\mathcal{C}(X) \equiv A^\top X + XA + Q + \Pi_1(X) - S(X)R(X)^{-1}S(X)^\top = 0, \tag{2.2}$$

with $R(X) \equiv R + \Pi_2(X) > 0$, $S(X) \equiv L + XB + \Pi_{12}(X)$ and the linear operator

$$\Pi(X) \equiv \begin{bmatrix} \Pi_1(X) & \Pi_{12}(X) \\ \Pi_{12}(X)^\top & \Pi_2(X) \end{bmatrix} \equiv \begin{bmatrix} A_0^\top X A_0 & A_0^\top X B_0 \\ B_0^\top X A_0 & B_0^\top X B_0 \end{bmatrix}.$$

The operator Π is positive as it satisfies $\Pi(X) \geq 0$ for $X \geq 0$. The optimal control is

$$u = -F(X)x, \quad F(X) \equiv R(X)^{-1}S(X)^\top = [R + \Pi_2(X)]^{-1}[L + XB + \Pi_{12}(X)]^\top,$$

with X being the maximal stabilizing solution of the CRRE (2.2). (Without loss of generality and for a simpler exposition, we include only one stochastic terms with one Wiener process ω_0; more terms in (2.1) only make Π in (2.2) a slightly more complicated expression with the same properties. We also assume R to be nonsingular, avoiding the generalized inverse for $R(X)$ in (2.2). We leave the general case for the future.)

In CRRE (2.2), it is convenient to write $R = S^\top S > 0$ for nonsingular matrix $S = R^{1/2}$. According to the condition measure in [18], the sensitivity of the solution of (2.2) to perturbations in S and C other than R and Q may be interesting. Also, perturbations may be limited by the psd structure of R and Q. Finally, the sensitivity of X with respect to perturbations of some underlying parameter set may be important.

Substitute $R = S^\top S$ in (2.2), we have

$$\begin{aligned} &S(X)R(X)^{-1}S(X)^\top = (L + XB + A_0^\top XB_0)(S^\top S + B_0^\top XB_0)^{-1}(L + XB + A_0^\top XB_0)^\top \\ =&(L + XB + A_0^\top XB_0)\left[S^\top(I_m + S^{-\top}B_0^\top XB_0S^{-1})S\right]^{-1}(L + XB + A_0^\top XB_0)^\top \\ =&(LS^{-1} + XBS^{-1} + A_0^\top XB_0S^{-1})(I_m + S^{-\top}B_0^\top XB_0S^{-1})^{-1}(LS^{-1} + XBS^{-1} + A_0^\top XB_0S^{-1})^\top, \end{aligned}$$

here B, B_0 and L are replaced by BS^{-1}, B_0S^{-1} and LS^{-1}, respectively, and the following condition is still satisfied

$$\begin{bmatrix} Q & LS^{-1} \\ S^{-\top}L^\top & I \end{bmatrix} = \begin{bmatrix} I & \\ & S^{-\top} \end{bmatrix} T \begin{bmatrix} I & \\ & S^{-1} \end{bmatrix} \geq 0.$$

Without loss of generality, we consider the case when $R = I_m$ and $Q = C^\top C \geq 0$, i.e., the rational Riccati equation

$$\begin{aligned} &A^\top X + XA + C^\top C + A_0^\top XA_0 \\ &\quad - (L + XB + A_0^\top XB_0)(I_m + B_0^\top XB_0)^{-1}(L + XB + A_0^\top XB_0)^\top = 0, \end{aligned} \tag{2.3}$$

with $I_m + B_0^\top XB_0 > 0$.

Suppose we introduce perturbations $\delta\Delta A$, $\delta\Delta B$, $\delta\Delta A_0$, $\delta\Delta B_0$, $\delta\Delta L$, $\delta\Delta C$ to the data matrices A, B, A_0, B_0, L, C respectively, the solution of the perturbed problem is $X + \delta\Delta X$ and δ is a small positive number, then the perturbed CRRE of (2.3) is

$$\begin{aligned} &(A + \delta\Delta A)^\top(X + \delta\Delta X) + (X + \delta\Delta X)(A + \delta\Delta A) + (C + \delta\Delta C)^\top(C + \delta\Delta C) \\ &+ (A_0 + \delta\Delta A_0)^\top(X + \delta\Delta X)(A_0 + \delta\Delta A_0) \\ &- \Big[(L + \delta\Delta L) + (X + \delta\Delta X)(B + \delta\Delta B) + (A_0 + \delta\Delta A_0)^\top(X + \delta\Delta X)(B_0 + \delta\Delta B_0)\Big] \\ &\Big[I_m + (B_0 + \delta\Delta B_0)^\top(X + \delta\Delta X)(B_0 + \delta\Delta B_0)\Big]^{-1} \\ &\Big[(L + \delta\Delta L) + (X + \delta\Delta X)(B + \delta\Delta B) + (A_0 + \delta\Delta A_0)^\top(X + \delta\Delta X)(B_0 + \delta\Delta B_0)\Big]^\top = 0. \end{aligned} \tag{2.4}$$

Dropping the second- and higher-order terms in (2.4), we obtain an equation in ΔX:

$$\begin{aligned}&\Phi^\top \Delta X + \Delta X \Phi + \Psi^\top \Delta X \Psi \\ &\approx -(\Delta C^\top C + C^\top \Delta C) - (\Delta A - \Delta B F)^\top X - X(\Delta A - \Delta B F) - (\Delta A_0 - \Delta B_0 F)^\top X \Psi \\ &- \Psi^\top X(\Delta A_0 - \Delta B_0 F) + \Delta L F + F^\top \Delta L^\top,\end{aligned} \tag{2.5}$$

where X is the exact solution of CRRE (2.3), $F = \left[I_m + B_0^\top X B_0\right]^{-1}\left[L + XB + A_0^\top X B_0\right]^\top$, $\Phi = A - BF$ and $\Psi = A_0 - B_0 F$. It is a generalized Lyapunov equation in ΔX. Applying the operator vec to both sides of the above equation by the identity $\mathrm{vec}(UVW) = (W^\top \otimes U)\mathrm{vec}(V)$, we obtain

$$\begin{aligned}Z\mathrm{vec}(\Delta X) &\approx -[(X \otimes I_n)\Upsilon + I_n \otimes X]\mathrm{vec}(\Delta A) + [(X \otimes F^\top)\Upsilon + F^\top \otimes X]\mathrm{vec}(\Delta B) \\ &- [((\Psi^\top X) \otimes I_n)\Upsilon + I_n \otimes (\Psi^\top X)]\mathrm{vec}(\Delta A_0) + [((\Psi^\top X) \otimes F^\top)\Upsilon + F^\top \otimes (\Psi^\top X)]\mathrm{vec}(\Delta B_0) \\ &+ [F^\top \otimes I_n + (I_n \otimes F^\top)\Upsilon]\mathrm{vec}(\Delta L) - [(C^\top \otimes I_n) + (I_n \otimes C^\top)\Upsilon]\mathrm{vec}(\Delta C^\top) \\ &= \{-[(X \otimes I_n)\Upsilon + I_n \otimes X], [(X \otimes F^\top)\Upsilon + F^\top \otimes X], \\ &- [((\Psi^\top X) \otimes I_n)\Upsilon + I_n \otimes (\Psi^\top X)], [((\Psi^\top X) \otimes F^\top)\Upsilon + F^\top \otimes (\Psi^\top X)], \\ &\quad [F^\top \otimes I_n + (I_n \otimes F^\top)\Upsilon], -[(C^\top \otimes I_n) + (I_n \otimes C^\top)\Upsilon]\} \\ &\times [\mathrm{vec}(\Delta A)^\top, \mathrm{vec}(\Delta B)^\top, \mathrm{vec}(\Delta A_0)^\top, \mathrm{vec}(\Delta B_0)^\top, \mathrm{vec}(\Delta L)^\top, \mathrm{vec}(\Delta C^\top)^\top]^\top, \\ &\equiv S_2 \mathrm{vec}([\Delta A, \Delta B, \Delta A_0, \Delta B_0, \Delta L, \Delta C^\top]),\end{aligned} \tag{2.6}$$

where $Z = I_n \otimes \Phi^\top + \Phi^\top \otimes I_n + \Psi^\top \otimes \Psi^\top$, S_2 is a matrix of size $n^2 \times p$ (with $p \equiv n(2n + 3m + l)$) and Υ is a permutation matrix given by $\Upsilon = \sum_{i,j} E_{ij} \otimes E_{ji}$, $E_{ij} = e_i e_j^\top$, e_i is the ith column of I_n and $\mathrm{vec}(A^\top) = \Upsilon \mathrm{vec}(A)$.

By the definition of absolute condition number [16, 25, 29], $\kappa(X) = \lim_{\epsilon \to 0} \sup_{\Delta \leq \epsilon} \|\Delta X\|_F / \epsilon$, with $\Delta = \left\| \left[\Delta A, \Delta B, \Delta A_0, \Delta B_0, \Delta L, \Delta C^\top\right] \right\|_F$, we get from (2.6) that

$$\kappa(X) \approx \|Z^{-1} S_2\|_F. \tag{2.7}$$

As shown in [20], we consider the componentwise condition numbers of X. The exact value of the condition number for the i-th component of $\mathrm{vec}(X)$ is

$$\kappa_i(X) \approx \|e_i^\top Z^{-1} S_2\|, \quad i = 1, \ldots, n^2, \tag{2.8}$$

where e_i is the i-th column of I_{n^2}.

The perturbation analysis of the rational (stochastic) Riccati equation has also been given in [4, 5], while we further exploit the psd structure in the perturbation of Q. The condition numbers $\kappa(X)$ and $\kappa_i(X)$ above, as well as the residual bounds in [4] and the condition numbers in [5] are all difficult to compute, especially for the large-scale problem, because of the large matrices from the Kronecker products. We adapt the SCE for the condition estimation.

3. Small Sample Statistical Condition Estimation

The ideas behind the SCE are well illustrated for functions $f : \mathbb{R}^p \to \mathbb{R}$ [18]. Assume that f is at least twice continuously differentiable. Local sensitivity can be measured by the norm of the gradient of f, $\nabla f(x) = (\frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_p})$. By Taylor expansion of f at a point $x \in \mathbb{R}^p$ along d, we have $f(x+\delta d) = f(x) + \delta \nabla f(x) d + O(\delta^2)$, where δ is a small positive number and $d \in \mathbb{R}^p$ has unit 2-norm. Then we get the inequality $|f(x+\delta d) - f(x)| \leq \delta \|\nabla f(x)\|$ up to first order in δ. This inequality points to the real utility of the local condition number $\|\nabla f(x)\|$, measuring the local sensitivity of f appropriately. In order to estimate the norm of $\|\nabla f(x)\|$, the quotient $|(f(x+\delta d) - f(x))/\delta|$ can be used to approximate the inner product $\nabla f(x) d$ between the gradient and the vector d. If d is selected uniformly and randomly from the unit p-sphere S_{p-1}, i.e., $d \in U(S_{p-1})$, then it is known (see discussion in [19]) that

$$\mathcal{E}(|\nabla f(x) d|) = \omega_p \|\nabla f(x)\|.$$

where the Wallis factor ω_p is defined by $\omega_1 = 1, \omega_2 = \frac{\pi}{2}$, and for $p > 2$,

$$\omega_p = \begin{cases} \dfrac{1 \cdot 3 \cdots (p-2)}{2 \cdot 4 \cdots (p-1)}, & \text{for } p \text{ odd}, \\ \dfrac{2}{\pi} \dfrac{2 \cdot 4 \cdots (p-1)}{1 \cdot 3 \cdots (p-2)}, & \text{for } p \text{ even}, \end{cases} \tag{3.1}$$

and the Wallis factor can be accurately approximated [19] by

$$\omega_p \approx \sqrt{\frac{2}{\pi(p - \frac{1}{2})}}. \tag{3.2}$$

It is shown in [19] that the condition estimator $\nu \equiv |\nabla f(x) d|/\omega_p$ satisfies $\mathcal{P}(\|\nabla f(x)\|/\gamma \leq \nu \leq \gamma \|\nabla f(x)\|) \geq 1 - \frac{2}{\pi\gamma} + \mathcal{O}(\frac{1}{\gamma^2})$, for $\gamma > 1$.

To improve the estimation procedure, we select $\bar{d}_1, \bar{d}_2, ..., \bar{d}_k \in \mathcal{N}(0,1)$ and then use a QR decomposition to produce an orthonormal basis $\{d_1, d_2, \ldots, d_k\}$ for their span $\mathcal{S}$, which is uniformly and randomly selected from the space of all k-dimensional subspaces of $\mathbb{R}^p$ [19]. Then the expected value of the norm of the projection of $\nabla f(x)$ onto $\mathcal{S}$ is given by

$$\mathcal{E}\left(\sqrt{|\nabla f(x) d_1|^2 + |\nabla f(x) d_2|^2 + \cdots + + |\nabla f(x) d_k|^2}\right) = \frac{\omega_p}{\omega_k} \|\nabla f(x)\|_2, \tag{3.3}$$

where ω_p and ω_k are Wallis factors with orders p and k, respectively. Then we get the subspace condition estimator

$$\nu(k) = \frac{\omega_k}{\omega_p} \sqrt{|\nabla f(x) d_1|^2 + |\nabla f(x) d_2|^2 + \cdots + + |\nabla f(x) d_k|^2} \tag{3.4}$$

has expected value $\|\nabla f(x)\|_2$. According to Theorem 3.3 of [19],

$$\begin{aligned} \mathcal{P}\left(\frac{\|\nabla f(x)\|}{\gamma} \leq \nu(2) \leq \gamma \|\nabla f(x)\|\right) &\approx 1 - \frac{\pi}{4\gamma^2}, \\ \mathcal{P}\left(\frac{\|\nabla f(x)\|}{\gamma} \leq \nu(3) \leq \gamma \|\nabla f(x)\|\right) &\approx 1 - \frac{32}{3\pi^2\gamma^3}, \\ \mathcal{P}\left(\frac{\|\nabla f(x)\|}{\gamma} \leq \nu(4) \leq \gamma \|\nabla f(x)\|\right) &\approx 1 - \frac{81\pi^2}{512\gamma^4}. \end{aligned} \tag{3.5}$$

These estimates are generally very accurate for $\gamma \geq 10$. This is the subspace statistical method and can give sharper estimates.

The function f may be scalar valued, but we can easily extend the SCE to twice continuously differentiable vector and matrix-valued functions. By using the vec operator [17], which maps matrices into vectors by stacking the columns, the set of vector-valued functions can be viewed as a map from $\mathbb{R}^p$ to $\mathbb{R}^q$ including the class of functions that map matrices into matrices [19]. Then the Taylor expansion about $x \in \mathbb{R}^p$ has the form

$$f(x + \delta d) = f(x) + \delta \nabla f(x) d + \mathcal{O}(\delta^2), \tag{3.6}$$

where $\nabla f(x)$ is a matrix of size $q \times p$ and d has unit norm. The sensitivity of f at x is measured by $\|\nabla f(x)\|$ to bound the perturbations in f and $\|\nabla f(x)\|$ is estimated by power method [25]. To be more precise, we can consider separately the sensitivity of every entry of f to the perturbations, especially since this entails no greater computational effort [19]. To be concrete,

$$f_i(x + \delta d) = f_i(x) + \delta \nabla f_i(x) d + \mathcal{O}(\delta^2), \quad i = 1, \ldots, q,$$

where f_i is the i-th entry of f and ∇f_i, the gradient of the scalar function f_i, is the i-the row of $\nabla f(x)$. Then $\|\nabla f_i\|$ can be estimated for $i = 1, \ldots, q$ by the theory of scalar functions. Furthermore, $\|\nabla f\|_F$ can be estimated.

From the point of view of Fréchet derivative, to measure the perturbations of the data matrices A, B, A_0, B_0, L and $C^\top$ (of the same dimension) in (2.3), we consider the mapping $g : \mathbb{R}^p \to \mathbb{R}^{n^2}$ with

$$\text{vec}(A, B, A_0, B_0, L, C^\top) \mapsto g(A, B, A_0, B_0, L, C^\top) \equiv \text{vec}(X),$$

where g is differentiable in a neighborhood of $(A, B, A_0, B_0, L, C^\top)$. Then we have the matrix $\mathcal{M}_{\nabla g}$ representing the Fréchet derivative $\nabla g(A, B, A_0, B_0, L, C^\top)$ on the direction $d \equiv \mathrm{vec}(\Delta A, \Delta B, \Delta A_0, \Delta B_0, \Delta L, \Delta C^\top)$ and satisfies

$$\mathrm{vec}(\Delta X) \approx \mathcal{M}_{\nabla g} d$$

with $\mathcal{M}_{\nabla g} = Z^{-1} S_2$ from (2.6). Then the absolute condition number (2.7) and the componentwise condition number (2.8) can be estimated by the SCE theory.

If $d \equiv \mathrm{vec}[\Delta A, \Delta B, \Delta A_0, \Delta B_0, \Delta L, \Delta C^\top]$ are selected uniformly and randomly from the unit p-sphere S_{p-1}, we get the solution ΔX by solving the generalized Lyapunov equation (2.5) and obtain the condition estimator $|\mathcal{M}_{\nabla g} d|/\omega_p$. Then the norm of i-th row of $\mathcal{M}_{\nabla g}$, i.e., the componentwise condition number $\kappa_i(X)$ is estimated by the i-th entry of $|\mathrm{vec}(\Delta X)|/\omega_p$; $\|\mathcal{M}_{\nabla g}\|_F$, i.e., the absolute condition number $\kappa(X)$ can be approximated by $\|\Delta X\|_F/\omega_p$. This is one sample condition estimation. For the subspace condition estimation, we state it in Algorithm 3.1 in detail.

Algorithm 3.1 (Subspace condition estimation for the solution X to CRRE (2.3)).

1. *Generate $(A^{(i)}, B^{(i)}, A_0^{(i)}, B_0^{(i)}, L^{(i)}, C^{(i)})$, $i = 1, \ldots, k$ with entries in $\mathcal{N}(0,1)$ and orthonormalize them. That is, with the QR factorization of the matrix*

$$\begin{bmatrix} \mathrm{vec}(A^{(1)}) & \cdots & \mathrm{vec}(A^{(k)}) \\ \vdots & & \vdots \\ \mathrm{vec}(C^{(1)}) & \cdots & \mathrm{vec}(C^{(k)}) \end{bmatrix},$$

   we get the the orthonormal matrix $\begin{bmatrix} q_1 & \cdots & q_k \end{bmatrix}$ and then convert each q_i back to the matrix $(A^{(i)}, B^{(i)}, A_0^{(i)}, B_0^{(i)}, L^{(i)}, C^{(i)})$ $i = 1, \ldots, k$ by the "unvec" operation. Here $A^{(i)}, A_0^{(i)} \in \mathbb{R}^{n \times n}$, $B^{(i)}, B_0^{(i)}, L^{(i)} \in \mathbb{R}^{n \times m}$ and $C^{(i)} \in \mathbb{R}^{l \times n}$.

2. *Let $p = (2n + 3m + l)n$. Approximate ω_p and ω_k using (3.2).*

3. *For $i = 1, \ldots k$, solve the generalized Lyapunov equation (2.5) to get the solution $Y_i = \Delta X$ with the matrix $(\Delta A, \Delta B, \Delta A_0, \Delta B_0, \Delta L, \Delta C^\top) = (A^{(i)}, B^{(i)}, A_0^{(i)}, B_0^{(i)}, L^{(i)}, C^{(i)\top})$.*

4. *Obtain the componentwise condition matrix*

$$\mathcal{K} = \frac{\omega_k}{\omega_p} \sqrt{|Y_1|^2 + \cdots + |Y_k|^2},$$

   then $\kappa_i(X)$ is estimated by the i-th entry of $\mathrm{vec}(\mathcal{K})$ and $\kappa(X)$ can be approximated by $\|\mathcal{K}\|_F$.

Similarly, for the relative condition number $\kappa^{\mathrm{R}}(X) = \lim_{\epsilon \to 0} \sup_{\Delta \le \epsilon} \|\Delta X\|_F/(\epsilon \|X\|_F)$, with $\Delta = \left\| \left[\frac{\Delta A}{\|A\|_F}, \frac{\Delta B}{\|B\|_F}, \frac{\Delta A_0}{\|A_0\|_F}, \frac{\Delta B_0}{\|B_0\|_F}, \frac{\Delta L}{\|L\|_F}, \frac{\Delta C^\top}{\|C\|_F} \right] \right\|_F$, we get from (2.6) that the relative condition number and componentwise relative condition number are

$$\kappa^{\mathrm{R}}(X) \approx \frac{\|Z^{-1} S_2 D_1\|_F}{\|X\|_F}, \quad \kappa_i^{\mathrm{R}}(X) \approx \begin{cases} \frac{\|e_i^\top Z^{-1} S_2 D_2\|_F}{|\mathrm{vec}(X)|_i}, & \text{if } |\mathrm{vec}(X)|_i \neq 0 \\ \|e_i^\top Z^{-1} S_2 D_2\|_F, & \text{if} |\mathrm{vec}(X)|_i = 0 \end{cases}, \quad i = 1, \ldots, n^2; \tag{3.7}$$
$$D_1 = \mathrm{diag}\{\|A\|_F I_{n^2}, \cdots, \|C\|_F I_{nl}\}, \quad D_2 = \mathrm{diag}\{|\mathrm{vec}(A)|, \cdots, |\mathrm{vec}(C^\top)|\}.$$

To relate this to the SCE technique, $\delta D_i d$ $(i = 1, 2)$ instead of δd describe the perturbations in (3.6), respectively, for d a unit-norm vector. In detail, for the relative condition number estimation, we *convert each $D_1 q_i$ back to the matrix $(A^{(i)}, B^{(i)}, A_0^{(i)}, B_0^{(i)}, L^{(i)}, C^{(i)})$ $i = 1, \ldots, k$ by the "unvec" operation* in step 1 in Algorithm 3.1 and $\kappa(X)$ *can be approximated by $\|\mathcal{K}\|_F/\|X\|_F$* in step 4; for the componentwise relative condition number estimation, we *convert each $D_2 q_i$ back to the matrix $(A^{(i)}, B^{(i)}, A_0^{(i)}, B_0^{(i)}, L^{(i)}, C^{(i)})$ $i = 1, \ldots, k$ by the "unvec" operation* in step 1 and $\kappa_i(X)$ *is estimated by the i-th entry of $\mathrm{vec}(\mathcal{K})$ divided componentwise by $|\mathrm{vec}(X)|$, leaving entries of $\mathrm{vec}(\mathcal{K})$ corresponding to zero entries of $|\mathrm{vec}(X)|$ unchanged* in step 4. The remaining steps in the algorithm are unchanged.

In practice, the exact Fréchet derivative cannot be obtained without the exact solution X. However, condition estimations usually need to be within only an order of magnitude, and this level of accuracy is attainable by an approximate solution [11].

	Componentwise				Normwise
Absolute Exact	5.4745	3.4747	3.4747	3.8951	8.3240
Absolute Estimated	5.4724	3.5270	3.5270	3.9842	8.5179
Relative Exact	1.5248	1.3927	1.3927	1.3997	2.9537
Relative Estimated	1.5507	1.4121	1.4121	1.4147	3.0541

Table 1: Componentwise and normwise condition estimations compared with the exact absolute and relative condition numbers.

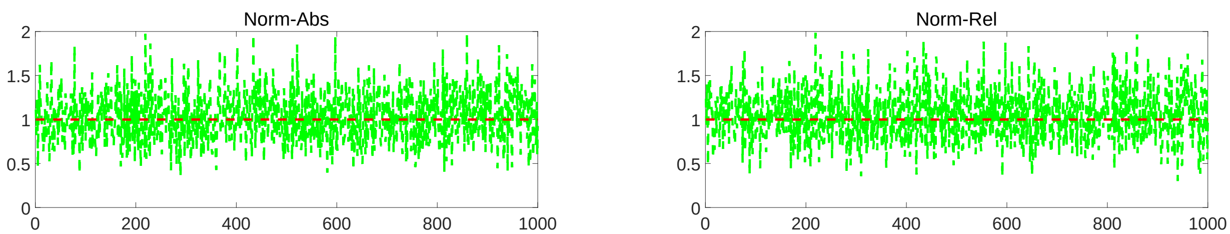
4. Numerical Examples

We present two examples to test the feasibility of the SCE for rational Riccati equations. The homotopy method is adopted for solving the rational Riccati equation to get an accurate approximate solution [28]. The core parts of the SCE as well as the homotopy method are solving the generalized Lyapunov equations [13]. All the examples have been attempted using MATLAB Ver. R2015b on a HUAWEI MateBook X Pro with an Intel Core i7 8550U CPU at 1.80GHz 1.99GHz and 16 GB.

Example 4.1. *Consider an example from [3, Example 1] to test the effects of the SCE, with the matrices*

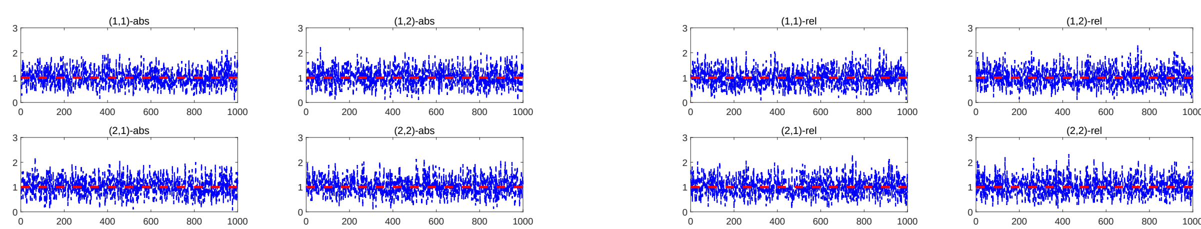
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix}, L = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, A_0 = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix}, B_0 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, R = 1.$$

We generate 1000 samples, then the normwise and componentwise condition estimations are shown in Figures 1,2 and the mean values of condition estimations are shown in Table 1, compared with the exact ones.



(a) norm absolute. (b) norm relative.

Figure 1: Normwise condition estimation.

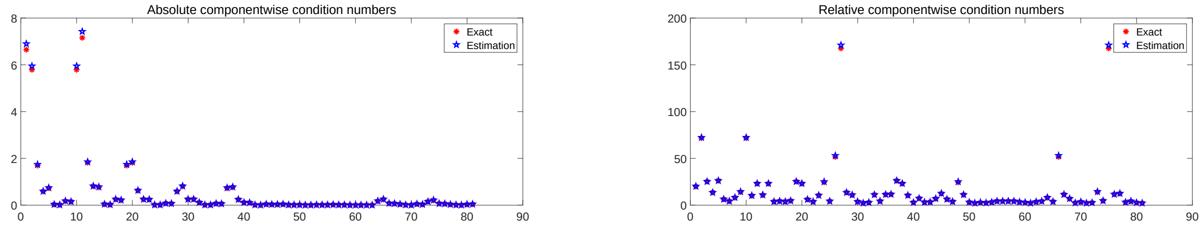


(a) norm absolute componentwise. (b) norm relative componentwise.

Figure 2: Componentwise condition estimation.

Example 4.2. *The example is from benchmark [3, Example 5] with the matrices* $A \in \mathbb{R}^{9\times 9}$, $B \in \mathbb{R}^{9\times 3}$, $C \in \mathbb{R}^{9\times 9}$ *and* $A_0 = 0.1 * I_9$, $B_0 = \begin{pmatrix} 0 \\ 0.1 * I_3 \end{pmatrix}$.

In this example, the componentwise condition numbers are much smaller than the normwise ones and may indicate the problem's true conditioning, shown in Figure 3.



(a) Exact:13.4235; Estimated: 14.0384.

(b) Exact:1.0136*1e3; Estimated: 1.0633*1e3.

Figure 3: Componentwise condition estimations show advantage over the normwise ones.

5. Conclusions

In this paper, we present the absolute and relative condition numbers for the rational Riccati equation, ensuring the symmetric positive semi-definition structure in some perturbations. Then we estimate the conditioning of the rational Riccati equations by the SCE techniques, considering normwise and componentwise perturbations. In numerical experiments, the SCE results agree with the exact values well. Thus the proposed algorithm is reliable and produce posterior error estimations of high accuracy. For the large scale problem, we can also get the condition estimations by modifying our algorithm and only need to consider the numerically low-rank and sparsity structures in solving the generalized Lyapunov equations. This will be our future work.

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