Calculus on Spaces with Higher Singularities

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Abstract. We establish extensions of the standard pseudo-differential calculus to specific classes of operators with operator-valued symbols occurring in symbolic hierarchies motivated by manifolds with higher singularities or stratified spaces.

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Introduction

Singular analysis is motivated by numerous and partly classical applications of physics and geometry, cf. [30], [24], [17], [18], see also the bibliographies there. The main intention of the present article is to give an idea on how algebras of pseudo-differential operators A can be organized when the underlying manifold M has singularities such as edges, boundaries or higher corners of some order. For a manifold with boundary the expectation is formally similar as in Boutet de Monvel's calculus, where the singular strata contribute an additional symbolic

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structure, in this case boundary symbols which contribute to ellipticity of boundary conditions, to parametrices of elliptic elements and to the Fredholm property when (in simplest cases) M is compact. It is natural to formulate similar structures in the case of manifolds $M \in \mathfrak{M}_k$ with higher singularities of order $k \in \mathbb{N}$, i.e., which are determined by a stratification

$$s(M) = \Big(s_0(M), s_1(M), \dots, s_k(M)\Big)$$

consisting of a sequence of smooth subspaces $s_j(M)$ for $\dim s_j(M) > \dim s_{j+1}(M)$ for $j = 0, \dots, k-1$. The principal symbolic structure of operators A is given by sequences of components

$$\sigma(A) = \Big(\sigma_0(A), \sigma_1(A), \dots, \sigma_k(A)\Big).$$

An example of a singular space is the infinite cone

$$X^\Delta := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X) \in \mathfrak{M}_1$$

for any $X \in \mathfrak{M}_0$; the case $k = 0$ indicates smoothness. In analysis over X^Δ we often pass to the corresponding open stretched cone

$$X^\wedge := \mathbb{R}_+ \times X$$

in a prescribed equivalence class of splitting of variables into $(r, x) \in \mathbb{R}_+ \times X$, representing regular singularity.

Ellipticity of an operator A over $M \in \mathfrak{M}_k$ is defined by some bijectivity conditions for the components of $\sigma(A)$. In general, those require operator block matrices $\mathcal{A} = (A_{ij})$ containing A in the upper left corner, up to Green and Mellin summands which are produced in compositions and parametrices. Those are related to the asymptotics of solutions as well as to the structure of extra trace and potential conditions coming from the lower-dimensional strata of M, see also [12].

This paper is organized as follows: The first Chapter is aimed at developing the approach for singularities of order $k = 1, 2$. We discuss a new intrinsic definition of Kegel spaces and complete the insight on corner pseudo-differential algebras up to singularity order 2. In addition we suggest a simplified approach into singular functions of asymptotics which belong to the tools of Green, trace and potential operators. Note that even the lower order singular analysis contains challenges and unexpected difficulties. For instance, an analogue of the Atiyah-Singer index theorem according to the K-theoretic approach of [2] or Boutet de Monvel [4] on operators with the transmission property seems to be not achieved yet. The multiplicative behaviour of singular spaces, appearing in Cartesian products $M^\Delta \times N^\Delta \in \mathfrak{M}_2$ for $M, N \in \mathfrak{M}_1$ is hard to complete by multiplicative properties of associated operators, say, in terms of Künneth formulas for elliptic complexes. Edge algebras and numerous structures in the context of boundary value problems are a special case of corner problems, while cone algebras, are a special case of the boundary symbolic calculus. Details can be found in Rempel, Schulze [45] ,[53],

[56] and in a series of joint papers jointly with Chang [8], [9], [10], as well in joint papers with Liu [35], Wei [61], Seiler [59], Hedayat-Mahmoudi [26].

The second Chapter is devoted to singular analysis of order $k \geq 3$. We show that our approach is iterative under repeatedly forming cones and wedges, starting with some base of given singularity order. Aspects on operators up to singularity order 2 have been studied in joint papers with Calvo [6], Maniccia [39], Habal [22], Rungrottheera and Wong [50]. Higher singular operators have been studied in [57], [58] and in joint papers of the second author with Calvo and Martin [5], Habal and Chang [7], Chang and Hedayat-Mahmoudi [11], Lyu [38].

1. Singular Manifolds and Corner-degenerate Differential Operators

1.1. Corner Manifolds

The underlying corner manifolds M of singularity order $k \in \mathbb{N}$, where $k = 0$ indicates smoothness, are defined as stratified spaces $M \in \mathfrak{M}_k$ such that for $k > 0$ there is an $s_k(M) \in \mathfrak{M}_0$, $s_k(M) \subseteq M$, such that $M \setminus s_k(M) \in \mathfrak{M}_{k-1}$, and there is a neighbourhood $V \subseteq M$ of $s_k(M)$ which has the structure of a locally trivial B^Δ_{k-1}-bundle over $s_k(M)$ for a compact $B_{k-1} \in \mathfrak{M}_{k-1}$ where $B^\Delta_{k-1} := (\overline{\mathbb{R}}_+ \times B_{k-1})/(\{0\} \times B_{k-1})$ is the infinite cone with base B_{k-1}, and the vertex $s_k(B^\Delta_{k-1})$ which is a single point. Often we briefly say that M is locally close to $s_k(M)$ modeled on $B^\Delta_{k-1} \times \mathbb{R}^{q_k}$ for $q_k := \dim\ s_k(M)$.

Note that $X \in \mathfrak{M}_0$ implies $X^\Delta \in \mathfrak{M}_1$, $X^\Delta \times X^\Delta \in \mathfrak{M}_2$, etc. More generally, $M \in \mathfrak{M}_k$, $\Omega \in \mathfrak{M}_0$ implies $M \times \Omega \in \mathfrak{M}_k$ and $\Omega = s_k(M \times \Omega) = s_k(M) \times \Omega$ which is in turn a special case of the rule $M \times N \in \mathfrak{M}_{k+l}$ for $M \in \mathfrak{M}_k$ for $N \in \mathfrak{M}_l$. We set $\dim\ M := \dim\ s_0(M)$ for any $M \times \Omega \in \mathfrak{M}_k$.

For an $M \in \mathfrak{M}_k$ the B^Δ_{k-1}-bundle over $s_k(M)$ represents a neighbourhood of $s_k(M)$ in M. On a manifold $M \in \mathfrak{M}_1$ with boundary $s_1(M)$ the situation is similar and quite common when we identify a collar neihgbourhood V of the boundary with the normal bundle with fibre $\overline{\mathbb{R}}_+$. In the general case, by replacing V by an $\overline{\mathbb{R}}_+ \times B_{k-1}$-bundle then the "bottom" $\{0\} \times B_{k-1}$ can be identified with an B_{k-1}-bundle which can be invariantly attached to $M \setminus s_k(M)$. Let $\mathbb{M}$ denote the resulting space. The double $2\mathbb{M}$ obtained by gluing together two copies of $\mathbb{M}$ by identifying the common subspace $s_k(M)$ then belongs to $\mathfrak{M}_{k-1}$. In order to illustrate the situation we consider the case $M = B^\Delta$ for a $B \in \mathfrak{M}_0$. Then B^Δ belongs to $\mathfrak{M}_1$, and we have $\mathbb{M} = \overline{\mathbb{R}}_+ \times B$, $2\mathbb{B} = \mathbb{R} \times B$.

Remark 1.1. *Observe that the strata $s_j(M)$ for $M \in \mathfrak{M}_k$ are not necessarily closed. For instance, if M is a disk in $\mathbb{C}$ then $s_0(M)$ is the open interior, and $s_1(M)$ the boundary. If $M \in \mathfrak{M}_k$ is embedded in $\mathbb{R}^n$ for some n and compact, then $s_k(M)$ is closed.*

For constructions below on compact $M \in \mathfrak{M}_k$ we choose specific partitions of unity. First there are functions $\psi_{k,l}$, $l = 1, \dots, N_k$ such that $\sum_{l=1}^{N_k} \psi_{k,l} = 1$ in a neighbourhood of $s_k(M)$. In addition we find functions $\psi_{k-1,l}$, $l = 1, \dots, N_{k-1}$ such that $\sum_{l=1}^{N_{k-1}} \psi_{k-1,l} = 1$ in a neighbourhood of $s_{k-1}(M)$. Moreover, there are functions $\psi_{k-2,l}$, $l = 1, \dots, N_{k-2}$ such that $\sum_{l=1}^{N_{k-2}} \psi_{k-2,l} = 1$ in a neighbourhood of $s_{k-2}(M)$. This process can be continued such that finally we obtain functions $\psi_{0,l}$, $l = 1, \dots, N_0$ such that $\sum_{l=1}^{N_0} \psi_{0,l} = 1$ in a neighbourhood of $s_0(M)$. We choose all these functions in a way that they restrict to element in $C^\infty(s_0(M))$. For $\Psi := \sum \psi_{m,l}$ with summation over all m, l we set $\varphi_{m,l} := \psi_{m,l}/\Psi$. Then $\sum_{m,l} \varphi_{m,l} = 1$ and $\sum_{l=1}^{N_m} \varphi_{m,l} = 1$ in a neighbourhood of $s_m(M)$ and the functions $\psi_{m,l}$ form the desired partition of unity.

1.2. Differential Operators and Principal Symbolic Hierarchies

We now have a look at typical differential operators on spaces $M \in \mathfrak{M}_k$. Let $\mathrm{Diff}^m(X)$ for any $X \in \mathfrak{M}_0$ be the Fréchet space of all differential operators on X of order m with smooth coefficients (in any local coordinates). An operator A on $s_0(M)$ for an $M \in \mathfrak{M}_k$, $k \geq 1$, is said to be an element in $\mathrm{Diff}^\mu_{\mathrm{deg}}(M)$ if in the case $\dim s_k(M) > 0$ it has the form

$$A = r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(r, y)\big(-r\frac{\partial}{\partial r}\big)^j (rD_y)^\alpha \tag{1.1}$$

for coefficients $a_{j\alpha}(r,y) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}_{\mathrm{deg}}(B_{k-1}))$, and in the case $\dim s_k(M) = 0$

$$A = r^{-\mu} \sum_{j=0}^{\mu} a_j(r)\big(-r\frac{\partial}{\partial r}\big)^j \tag{1.2}$$

for coefficients $a_j(r) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-j}_{\mathrm{deg}}(B_{k-1}))$. Clearly, deg can be omitted for $k - 1 = 0$. As in the preceding subsection, $M \in \mathfrak{M}_k$ is locally close to $s_k(M)$ modeled on $X^\Delta_{k-1} \times s_k(M)$ and the Fréchet spaces $\mathrm{Diff}^\nu(X_{k-1})$ are defined by the iterative step before, where $\mathrm{Diff}^\nu(X)$ for $X \in \mathfrak{M}_0$ is the standard space of differential operators with smooth coefficients. For A given by (1.1) and (1.2) we set

$$\sigma_k(A)(y, \eta) = r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(0, y)\big(-r\frac{\partial}{\partial r}\big)^j (r\eta)^\alpha \tag{1.3}$$

and

$$\sigma_k(A)(v) = \sum_{j=0}^{\mu} a_j(0) v^j, \tag{1.4}$$

respectively. In (1.3) we assume $\eta \neq 0$. In (1.4) v is interpreted as a complex Mellin covariable, often regarded as an element of

$$\Gamma_\lambda := \{v \in \mathbb{C} : \operatorname{Re} v = \lambda\}$$

for a suitable real λ, where λ is associated to a corresponding weight. For $A \in \mathrm{Diff}^\mu(M)$, $B \in \mathrm{Diff}^\nu(M)$, $M \in \mathfrak{M}_k$, and $\dim s_k(M) > 0$ we have $\sigma_k(AB)(y,\eta) = \sigma_k(A)(y,\eta)\sigma_k(B)(y,\eta)$. In the case $\dim s_k(M) = 0$ we have $\sigma_k(AB)(v) = \sigma_k(A)(v-\nu)\sigma_k(B)(v)$. For $A \in \mathrm{Diff}^\mu_{\deg}(M)$, $k \geq 1$, we have at the same time $A \in \mathrm{Diff}^\mu_{\deg}(M \setminus s_k(M))$. Since $M \setminus s_k(M) \in \mathfrak{M}_{k-1}$ we can determine $\sigma_{k-1}(A)$, where we always have $\dim s_{k-1}(M) > 0$. Thus we get the full principal symbolic hierarchy $\sigma(A)$.

Example 1. Operators of the form

$$A := (r_1 \cdots r_k)^{-\mu}(r_1\partial_{r_1})^{j_1}(r_1r_2\partial_{r_2})^{j_2} \cdots (r_1r_2 \cdots r_k\partial^{j_k}_{r_k})$$

for $j_1 + \cdots + j_k = k$ belong to $\mathrm{Diff}^\mu_{\deg}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \cdots \times \overline{\mathbb{R}}_+)$ over $M := \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \cdots \times \overline{\mathbb{R}}_+ \in \mathfrak{M}_k$.

1.3. Operators for Singularities of Lower Order

By singularities of lower order we understand the cases $k \leq 2$ which are necessary for the iterative process. We first look at the case $\dim(s_k(M)) > 0$. Basics for $k = 2$ are developed in the above-mentioned articles, especially in [26] T.he case $\dim(s_k(M)) = 0$ is a slight modification. The operator-valued symbols (1.4) induce families of continuous operators between Kegel spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$

$$\sigma_k(A)(y,\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \tag{1.5}$$

for $s, \gamma \in \mathbb{R}$ when $X \in \mathfrak{M}_0$ and $k = 1$ and for $s \in \mathbb{R}$, $\gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2$ when $X \in \mathfrak{M}_1$ and $k = 2$. In this case we define $\gamma - \mu := (\gamma_i - \mu)_{i=1,2}$. While Kegel spaces for manifolds $M \in \mathfrak{M}_1$ with edge are a traditional tool in singular analysis the definition for $X \in \mathfrak{M}_1$ is by no means straightforward. Let us briefly recall some notions in this context. We systematically employ the weighted Mellin transform of weight $\gamma \in \mathbb{R}$

$$M_\gamma u(v) := \int_0^\infty r^v u(r')dr'/r'$$

first regarded as a continuous map $M_\gamma : r^\gamma L^2(\mathbb{R}_+) \to L^2(\Gamma_{1/2-\gamma})$ where spaces on Γ_λ are interpreted as the standard ones with respect to the real variable $\operatorname{Re} v$ for $v \in \Gamma_\lambda$. We then have weighted Mellin pseudo-differential operators on the r half-axis

$$\mathrm{Op}^\gamma_M(f)u := M^{-1}_{\gamma,v\to r} f(r,r',v) M_{\gamma,r'\to v} u \tag{1.6}$$

for any $f(r,r',v) \in S^\mu(\mathbb{R}_+ \times \mathbb{R}_+ \times \Gamma_{1/2-\gamma})$ in Hörmander's symbol class, later on to be generalized to operator-valued symbols, or, equivalently,

$$\mathrm{Op}^\gamma_M(f)u(r) = \iint \left(r/r'\right)^{-(1/2-\gamma+i\rho)} f(r,r',1/2-\gamma+i\rho)u(r')dr'/r'đ\rho \tag{1.7}$$

for $đ\rho = (2\pi i)^{-1}d\rho$.

A separable Hilbert space H is said to be endowed with a group action $\kappa = \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$ if $\kappa_\delta : H \to H$ is an isomorphism for every δ, moreover, $\kappa_\delta \kappa_{\delta'} = \kappa_{\delta\delta'}$ for every $\delta, \delta' \in \mathbb{R}_+$, and $h \mapsto \kappa_\delta h$ represents an element $f(\delta) \in C(\mathbb{R}_+, H)$ for every $h \in H$. A Fréchet space E is said to be endowed with a group action κ, if $E = \varprojlim_{j \in \mathbb{N}} E^j$ is written as a projective limit of Hilbert spaces E^j with continuous embeddings $E^j \hookrightarrow E^0$ for all j, where E^0 is endowed with $\kappa = \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$ as well as all E^j with $\kappa_\delta|_{E^j}$ for all j.

For two Hilbert spaces H and $\tilde{H}$ with group action κ and $\tilde{\kappa}$, respectively, $\Omega \subseteq \mathbb{R}^p$ open, we have the spaces

$$S^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H})$$

of operator-valued symbols of order $\mu \in \mathbb{R}$, consisting of those $a(y, \eta) \in C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(H, \tilde{H}))$, satisfying the "twisted" symbolic estimates

$$\|\tilde{\kappa}^{-1}_{\langle\eta\rangle} \{D^\alpha_y D^\beta_\eta a(y, \eta)\} \kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H, \tilde{H})} \leq c \langle\eta\rangle^{\mu - |\beta|}$$

for all $(y, \eta) \in K \times \mathbb{R}^q, K$ compact in Ω, all $\alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^q$, for constants $c = c(\alpha, \beta, K) > 0$. A similar definition works for classical symbols with components of twisted homogeneity $\mu - j$, $j \in \mathbb{N}$. Similarly, we form symbol spaces $S^\mu(\Omega \times \mathbb{R}^q; E, \tilde{E})$ for Fréchet spaces E, $\tilde{E}$ with group action. We tacitly use this material below and refer to the monographs [53], [56] or [15].

Let H be a separable Hilbert space with group action $\kappa = \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$. Then $\mathcal{W}^s(\mathbb{R}^q, H)$ for $s \in \mathbb{R}$ is defined as the completion of $\mathcal{S}(\mathbb{R}^q, H)$ with respect to the norm

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q, H)} = \Big\{ \int \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle} (Fu)(\eta)\|^2_H \,đ\eta \Big\}^{1/2} \tag{1.8}$$

for $đ\eta = (2\pi)^{-q} d\eta$. The definition extends to the case of a smooth manifold Y rather than $\mathbb{R}^q$, and for non-compact Y we also have "comp"/"loc"-versions of such spaces. Another generalization concerns Fréchet spaces E instead of Hilbert spaces, written as projective limits of Hilbert spaces with group action. We also employ such generalizations.

Given a Hilbert space H with group action $\kappa = \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$ we have local weighted Mellin Sobolev spaces $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^q, H)$, defined as completion of $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^q, H)$ with respect to the norm

$$\Big\{ \int_{\mathbb{R}^q} \int_{\Gamma_{(b+1)/2-\gamma}} \langle v, \eta\rangle^{2s} \|\kappa^{-1}_{\langle v, \eta\rangle} M_{r \to v} F_{y \to \eta}(u)(v, \eta)\|^2_H \,đv đ\eta \Big\}^{1/2}. \tag{1.9}$$

The number b is fixed in connection with extra information from the space H, for instance, $b := n$ in the case $\mathcal{K}^{s,\gamma}(X^\wedge)$ for $X \in \mathfrak{M}_0$, $n = \dim X$ which is endowed with the group action

$$(\kappa_\delta u)(r, x) = \delta^{(n+1)/2} u(\delta r, x). \tag{1.10}$$

For $q = 0$, $H = \mathbb{C}$ and $\kappa_\delta = \mathrm{id}_H$, $\delta \in \mathbb{R}_+$, we have the spaces $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ as well as $\mathcal{H}^{s,\gamma}(X^\wedge)$. The latter ones are obtained for $X \in \mathfrak{M}_0$, $n := b := \dim X$ and $H := \mathbb{C}$ by completing $C_0^\infty(X^\wedge)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} := \Big\{ \sum_{j=0}^{N} \|\varphi_j u \circ (\mathrm{id}_{\mathbb{R}_+} \times \chi_j^{-1})_*)\|^2_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)} \Big\}^{1/2}$$

where $(U_1, \ldots, U_N)$ is a covering of X by coordinate neighbourhoods, $(\varphi_1, \ldots, \varphi_N)$ a subordinate partition of unity, $\chi_j : U_j \to \mathbb{R}^n$ charts, moreover, $(\cdot)_*$ indicates the push forward under the respective diffeomorphism, and $\mathrm{id}_{\mathbb{R}_+}$ denotes the identity diffeomorphism on $\mathbb{R}_+$. For $X \in \mathfrak{M}_0$ a possible definition of the Kegel space $\mathcal{K}^{s,\gamma}(X^\wedge)$ is

$$\mathcal{K}^{s,\gamma}(X^\wedge) = \{\omega u + (1-\omega)v : u \in \mathcal{H}^{s,\gamma}(X^\wedge), v \in H^s_{\mathrm{cone}}(X^\wedge)\}, \tag{1.11}$$

where $H^s_{\mathrm{cone}}(X^\wedge)$ is modeled for $r \to \infty$ on standard Sobolev spaces on $\mathbb{R}^{n+1}$ of smoothness s in coordinates $\tilde{x}$ over $\mathbb{R}_+ \times U$ for any coordinate neighbourhood U on X diffeomorphic to the open unit ball in $\mathbb{R}^n$ under the identification $\mathbb{R}_+ \times U \to \{\text{cone}\}$, where "cone" indicates a conical subset in $\mathbb{R}^{n+1} \setminus \{0\}$ obtained by $\{\tilde{x} = (\tilde{x}_0, r\tilde{x}') : \tilde{x}_0 = r, \tilde{x}' := (\tilde{x}_1, \ldots, \tilde{x}_n)\}$ for $\tilde{x}'$ in the above-mentioned open unit ball in $\mathbb{R}^n$. The definition of $\mathcal{K}^{s,\gamma}(X^\wedge)$ does not depend on the choice of the cut-off function ω, because of $\mathcal{H}^{s,\gamma}(X^\wedge)$, $H^s_{\mathrm{cone}}(X^\wedge) \subseteq H^s_{\mathrm{loc}}(X^\wedge)$.

We have

$$\mathcal{K}^{0,0}(X^\wedge) = \mathcal{H}^{0,0}(X^\wedge) = r^{-n/2}L^2(X^\wedge), \tag{1.12}$$

and (1.10) is unitary in $r^{-n/2}L^2(X^\wedge)$. Moreover, if $\mathbb{K}^\beta(r)$ is any strictly positive smooth function on $\mathbb{R}_+$ that is equal to 1 for $r > \varepsilon_1$ and r^β for $0 < r < \varepsilon_0$ for some $0 < \varepsilon_0 < \varepsilon_1 < \infty$ then we have the relation

$$\mathcal{K}^{s,\gamma}(X^\wedge) = \mathbb{K}^\gamma \mathcal{K}^{s,0}(X^\wedge) \tag{1.13}$$

where $\mathbb{K}^\gamma$ is regarded as operator of multiplication by the corresponding function.

There is another "intrinsic" definition of Kegel spaces which is apparently more natural in analogous form for X of higher singular order. We therefore sketch more tools from the edge calculus of singularity order 1. If X is a smooth closed manifold of dimension n then $L^\mu_{(\mathrm{cl})}(X; \mathbb{R}^d)$ means the space of classical or non-classical (indicated by subscript "(cl)") parameter-dependent pseudo-differential operators on X of order $\mu \in \mathbb{R}$ in its natural Fréchet topology. Observe, in particular, that $L^{-\infty}(X; \mathbb{R}^d) = \bigcap_{\mu \in \mathbb{R}} L^\mu(X; \mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d, L^{-\infty}(X))$ with $L^{-\infty}(X)$ being identified with $C^\infty(X \times X)$ via a fixed Riemannian metric on X. Let E be a Fréchet space. By $\mathcal{A}(U, E)$ for an open subset $U \subseteq \mathbb{C}$ we denote the space of all holomorphic functions in U with values in E in the Fréchet topology of uniform convergence on compact subsets of U.

Definition 1.2. Let $M_{\mathcal{O}}^{\mu}(X;\mathbb{R}^d)$ denote the space of all $h(v,\zeta) \in \mathcal{A}(\mathbb{C}_v, L_{\mathrm{cl}}^{\mu}(X;\mathbb{R}_{\zeta}^d))$ such that $h(\lambda + i\rho, \zeta) \in L_{\mathrm{cl}}^{\mu}(X;\Gamma_\lambda \times \mathbb{R}_{\zeta}^d)$ for every $\lambda \in \mathbb{R}$, uniformly in compact λ-intervals.

The alternative definition of Kegel spaces is due to the following result of the edge calulus, cf. [49], [38]. We set

$$M_{\mathcal{O}}^{\mu}(X;\mathbb{R}_{r\eta}^q) := \{\tilde{h}(v,r\eta) : \tilde{h}(v,\tilde{\eta}) \in M_{\mathcal{O}}^{\mu}(X;\mathbb{R}_{\tilde{\eta}}^q)\}. \tag{1.14}$$

Elements of (1.14) are regarded as Mellin symbols, i.e., symbols of Mellin pseudo-differential operators

$$\mathrm{Op}_M^{\gamma}(h)(\eta)u(r) = \int\int (r/r')^{-(1/2-\gamma+i\rho)} h(r, 1/2 - \gamma + i\rho, \eta) u(r') dr'/r' đ\rho$$

for $h(r,v,\eta) \in M_{\mathcal{O}}^{\mu}(X;\mathbb{R}_{r\eta}^q)$, $đ\rho = (2\pi)^{-1}d\rho$. The spaces $M_{\mathcal{O}}^{\mu}(X;\mathbb{R}_{\zeta}^d)$ have some natural properties: First kernel cut-off gives us a continuous map

$$L_{\mathrm{cl}}^{\mu}(X;\Gamma_\lambda \times \mathbb{R}_{\zeta}^d) \to M_{\mathcal{O}}^{\mu}(X;\mathbb{R}_{\zeta}^d).$$

The assertions of the following theorem are well-known, cf. [53] or [56] and valid in analogous form for holomorphic symbols in connection with singularities of higher order.

Theorem 1.3. (i) *Let $h(v,\zeta) \in M_{\mathcal{O}}^{\mu}(X;\mathbb{R}_{\zeta}^d)$, $h|_{\Gamma_\lambda} \in L_{\mathrm{cl}}^{\mu-1}(X;\Gamma_\lambda \times \mathbb{R}_{\zeta}^d)$ for some λ. Then $h \in M_{\mathcal{O}}^{\mu-1}(X;\mathbb{R}_{\zeta}^d)$.*

(ii) *$h(v,\zeta) \in M_{\mathcal{O}}^{\mu}(X;\mathbb{R}_{\zeta}^d)$ entails $(T^\beta h)(v,\zeta) := h(v+\beta,\zeta) \in M_{\mathcal{O}}^{\mu}(X;\mathbb{R}_{\zeta}^d)$ for every $\beta \in \mathbb{R}$.*

(iii) *For $h(v,\zeta) \in M_{\mathcal{O}}^{\mu}(X;\mathbb{R}_{\zeta}^d)$ we have $(\partial_v h)(v,\zeta) \in M_{\mathcal{O}}^{\mu}(X;\mathbb{R}_{\zeta}^d)$.*

(iv) *For $h(v,\zeta) \in M_{\mathcal{O}}^{\mu}(X;\mathbb{R}_{\zeta}^d)$ and every δ, γ we have $\mathrm{Op}_M^{\delta}(h)(\zeta) = \mathrm{Op}_M^{\gamma}(h)(\zeta)$ on functions in $C_0^\infty(\mathbb{R}_+)$.*

(v) *For $h(v,\zeta) \in M_{\mathcal{O}}^{\mu}(X;\mathbb{R}_{\zeta}^d)$ we have $\mathrm{Op}_M^{\gamma}(h)(\zeta) r^\alpha = r^\alpha \mathrm{Op}_M^{\gamma}(T^{-\alpha}h)(\zeta)$ for every α, γ on $C_0^\infty(\mathbb{R}_+)$, $(T^{-\alpha}h)(v,\zeta) = h(v-\alpha,\zeta)$.*

It is useful to enrich the space $M_{\mathcal{O}}^{\mu}(X;\mathbb{R}_{\tilde{\eta}}^q)$ by an extra parameter $\iota \in \mathbb{R}^w$ for some $w \in \mathbb{N}$ by replacing $L_{\mathrm{cl}}^{\mu}(X;\mathbb{R}_{\tilde{\eta}}^q)$ by $L_{\mathrm{cl}}^{\mu}(X;\mathbb{R}_\iota^w \times \mathbb{R}_{\tilde{\eta}}^q)$. Let us denote the corresponding class by

$$M_{\mathcal{O}}^{\mu}(X;\mathbb{R}_\iota^w \times \mathbb{R}_{r\eta}^q). \tag{1.15}$$

Clearly we have

$$M_{\mathcal{O}}^{\mu}(X;\mathbb{R}_\iota^w \times \mathbb{R}_{r\eta}^q) \subset M_{\mathcal{O}}^{\mu}(X;\mathbb{R}_{r\eta}^q)$$

for every fixed $\iota \in \mathbb{R}^w$. Therefore, we often suppress the parameter ι though it could be added in many relations.

We need some observations on associated operators, similarly as technique from Kumano-go' calculus [33], or of Seiler [66]. Let

$$f(v,r\eta)\in M^{\mu}_{\mathcal{O}}(X;\mathbb{R}^{w}_{\iota}\times\mathbb{R}^{q}_{r\eta}),\ g(v,r\eta)\in M^{\nu}_{\mathcal{O}}(X;\mathbb{R}^{w}_{\iota}\times\mathbb{R}^{q}_{r\eta}),$$

and set

$$A:=r^{-\mu}\mathrm{Op}_M^{\gamma-\nu-n/2}(f)(\eta),\ B:=r^{-\nu}\mathrm{Op}_M^{\gamma-n/2}(g)(\eta) \tag{1.16}$$

and form in admitted manner (since f, g are holomorphic in v) the composition

$$AB=r^{-\mu}\mathrm{Op}_M^{\gamma-\nu-n/2}(f)(\eta)\circ r^{-\nu}\mathrm{Op}_M^{\gamma-n/2}(g)(\eta). \tag{1.17}$$

Then

$$AB=r^{-(\mu+\nu)}\mathrm{Op}_M^{\gamma-n/2}(h\#g) \tag{1.18}$$

for $h(v,r\eta)=f(v-\nu,r\eta)$ and

$$h\#g\sim\sum_{k=0}^{N}\frac{1}{k!}\partial_v^k h\big(-r\partial_r\big)^k g+r_{N+1} \tag{1.19}$$

where the remainder is a Mellin oscillatory integral
(1.20)

$$r_{N+1}=\int_0^1\frac{(1-\theta)^N}{N!}\iint t^{i\tau}\partial_v^{N+1}h(v+i\theta\tau,r\eta)\big(-r\partial_r\big)^{N+1}g(v,tr\eta)\frac{dt}{t}đ\tau d\theta.$$

We obtain

$$r_{N+1}(v,r\eta)\in M_{\mathcal{O}}^{\mu+\nu-(N+1)}(X;\mathbb{R}^q_{r\eta}) \tag{1.21}$$

and hence

Theorem 1.4. *$h\in M^{\mu}_{\mathcal{O}}(X;\mathbb{R}^{w}_{\iota}\times\mathbb{R}^{q}_{r\eta})$, $g\in M^{\nu}_{\mathcal{O}}(X;\mathbb{R}^{w}_{\iota}\times\mathbb{R}^{q}_{r\eta})$ entails $h\#g\in M^{\mu+\nu}_{\mathcal{O}}(X;\mathbb{R}^{w}_{\iota}\times\mathbb{R}^{q}_{r\eta})$.*

The following result is known from [66] or [20], but our proof also applies in more complicated situations below. Since $q\in\mathbb{N}$ is arbitrary, we also may replace f, g by

$$f(r,v,\eta,\zeta),\ g(r,v,\eta,\zeta)\in M^{\mu}_{\mathcal{O}}(X;\mathbb{R}^{q+d}_{r\eta,r\zeta})$$

for an additional parameter $\zeta\in\mathbb{R}^d$ for any $d\in\mathbb{N}$.

Theorem 1.5. *For every $\mu,\gamma\in\mathbb{R}$ there exists an element $f^{\mu}(r,v,\eta)\in M^{\mu}_{\mathcal{O}}(X;\mathbb{R}^{q}_{r\eta})$ such that for any sufficiently large $|\eta|$ the operator*

$$r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(f^{\mu})(\eta):\mathcal{K}^{s,\gamma}(X^{\wedge})\to\mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge}) \tag{1.22}$$

is an isomorphism for every $s\in\mathbb{R}$.

Proof. We choose a parameter-dependent elliptic $\tilde{l}(\iota,\rho,\tilde{\eta})\in L^{\mu}_{\mathrm{cl}}(X;\mathbb{R}^{w}_{\iota}\times\Gamma_{\lambda}\times\mathbb{R}^{q}_{\tilde{\eta}})$ and produce via kernel cut-off turning ρ on $\Gamma_{\lambda}$to the complex variable v an element $\tilde{f}(\iota,v,\tilde{\eta})\in M^{\mu}_{\mathcal{O}}(X;\mathbb{R}^{w}_{\iota}\times\mathbb{R}^{q}_{\tilde{\eta}})$. Then, setting $f^{\mu}(r,v,\eta):=\tilde{f}(\iota,v,r\eta)$ we first see that (1.22) is an elliptic operator in the cone algebra over $X^{\wedge}$ with conormal symbol

$$f^{\mu}(\iota,v,0):H^{s}(X)\to H^{s-\mu}(X) \tag{1.23}$$

which is an elliptic pseudo-differential operator with parameter $(\iota, v) \in \mathbb{R}^w_\iota \times \Gamma_{(n+1)/2-\gamma}$ and bijective for sufficiently large $|\iota|$. If necessary, we modify the Mellin symbol vy a translation in the complex v-plane. At the same time (1.22) is an edge symbol in its dependence on η which is homogeneous in the sense

$$r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(f^\mu)(\delta\eta) = \delta^\mu \kappa_\delta \big(r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(f^\mu)(\eta)\big)\kappa_\delta^{-1}. \tag{1.24}$$

Composing $r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(f^\mu)(\eta)$ from the right with $r^\mu \mathrm{Op}_M^{\gamma-\mu-n/2}((f^\mu)^{-1})(\eta)$ we can first commute r^μ in the middle through the Mellin action, cf. Theorem 1.3 (iv), and then apply the above-mentioned composition result. Then, for sufficiently large $\iota, |\eta|$ the remainder becomes small and hence we can compute the inverse of $r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(f^\mu)(\eta)$. □

Thus, setting $R^{-s}(\eta) := r^s \mathrm{Op}_M^{\gamma+s-n/2}(f^{-s})(\eta)$ for suitable $f^{-s}(\eta) \in M_{\mathcal{O}}^{-s}(X; \mathbb{R}^q_{r\eta})$ we can define

$$\mathcal{K}^{s,\gamma}(X^\wedge) = R^{-s}(\eta)\mathcal{K}^{0,\gamma-s}(X^\wedge) \tag{1.25}$$

for any $|\eta|$ sufficiently large.

Kegel spaces give rise to edge spaces, cf. relation (1.8) for $H = \mathcal{K}^{s,\gamma}(X^\wedge)$.

For singularity order 2 we now consider a compact element $B \in \mathfrak{M}_1$ with edge $Y := s_1(B)$, $\dim Y =: q > 0$, and define spaces $\mathcal{K}^{s,\gamma}(B^\wedge)$ for $s \in \mathbb{R}$ and a pair of weights $\gamma := (\gamma_1, \gamma_2) \in \mathbb{R}^2$. Assume, for convenience, that Y has a neighbourhood V which corresponds to a trivial X^Δ-bundle over Y for some compact $X \in \mathfrak{M}_0$. On our $B \in \mathfrak{M}_1$ with edge Y, locally near the edge modeled on $X^\Delta \times Y$ we first form the edge space $\mathcal{W}^s(Y, \mathcal{K}^{s,\gamma_1}(X^\wedge))$ for a weight $\gamma_1 \in \mathbb{R}$ and define

$$H^{s,\gamma_1}(B) := \omega_1 \mathcal{W}^s(Y, \mathcal{K}^{s,\gamma_1}(X^\wedge)) + (1-\omega_1)H^s(2\mathbb{B}) \tag{1.26}$$

for any cut-off function $\omega_1 := \omega_1(r_1)$ on B, i.e., some function in $C^\infty(s_0(B))$ which is equal to 1 in a neighbourhood of Y. Set $H^{\infty,\gamma_1}(B) := \bigcap_{s\in\mathbb{R}} H^{s,\gamma_1}(B)$.

Let

$$\begin{aligned} \mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B)) := {} & \omega_2 \mathcal{H}^{\infty,\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B)) \\ & + (1-\omega_2)\mathcal{S}(\mathbb{R}, H^{\infty,\gamma_1}(B))|_{\mathbb{R}_+}, \end{aligned} \tag{1.27}$$

where $\omega_2 := \omega_2(r_2)$ is a cut-off function on the r_2 half-axis. We form the space of edge pseudo-differential operators

$$L^\mu(B, \boldsymbol{g}_B; \mathbb{R}^d) \subset L^\mu_{\mathrm{cl}}(s_0(B; \mathbb{R}^d) \tag{1.28}$$

for weight data $\boldsymbol{g}_B := (\gamma_1, \gamma_1 - \mu, \Theta_1)$, with $\Theta_1 := (-(\vartheta_1 + 1), 0]$ for a $\vartheta_1 \in \mathbb{N}$.

Definition 1.6. Let $M^\mu_{\mathcal{O}}(B, \boldsymbol{g}_B; \mathbb{R}^d)$ denote the space of all

$$h(v_2, \zeta) \in \mathcal{A}(\mathbb{C}_{v_2}, L^\mu(B, \boldsymbol{g}_B; \mathbb{R}^d_\zeta)))$$

such that $h(\lambda + i\rho, \zeta) \in L^\mu(B, \boldsymbol{g}_B; \Gamma_\lambda \times \mathbb{R}^d)$ for every $\lambda \in \mathbb{R}$, uniformly in compact λ-intervals.

Definition 1.7. For compact $B \in \mathfrak{M}_1$ with edge Y, locally near Y modeled on $X^\Delta \times Y$ for an $X \in \mathfrak{M}_0$ and $\gamma := (\gamma_1, \gamma_2)$ we define

(1.29)
$$\mathcal{K}^{0,\gamma}(B^\wedge) := \omega_2\omega_1\mathcal{H}^{0,\gamma_2}(\mathbb{R}_+ \times Y, \mathcal{K}^{0,\gamma_1}(X^\wedge)) + (1-\omega_2)\omega_1\mathcal{H}^{0,0}(\mathbb{R}_+ \times Y, \mathcal{K}^{0,\gamma_1}(X^\wedge)) \\ + (1-\omega_2)(1-\omega_1)\mathcal{K}^{0,0}((2\mathbb{B})^\wedge)) + \omega_2(1-\omega_1)\mathcal{K}^{0,\gamma_2}((2\mathbb{B})^\wedge))$$

for cut-off functions $\omega_i = \omega_i(r_i)\, i = 1.2$.

In the latter definition we employ that the double $2\mathbb{B}$ belongs to $\mathfrak{M}_0$. The spaces of Definition 1.7 are Hilbert with scalar products from the non-direct sum. In particular, we fix the scalar product of $\mathcal{K}^{0,0}(B^\wedge)$ as a reference scalar product.

Note that $\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ defined by

$$(\kappa_\delta u)(r_k, \cdot) = \delta^{(b+1)/2} u(\delta r_k, \cdot) \tag{1.30}$$

turns $\mathcal{K}^{0,\gamma}(B^wedge)$ to a Hilbert space with group action.

We have a non-degenerate sesquilinear pairing

$$(\cdot,\cdot)_{\mathcal{K}^{0,0}(B^\wedge)} : \mathcal{K}^{0,\gamma}(B^\wedge) \times \mathcal{K}^{0,-\gamma}(B^\wedge) \to \mathbb{C} \tag{1.31}$$

for any $\gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2$, and we have a natural inclusion

$$\mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B)) \subseteq \mathcal{K}^{0;\gamma_2,\gamma_2}(B^\wedge). \tag{1.32}$$

This allows us to identify the space

$$\mathcal{S}^{-\gamma_2}(\mathbb{R}_+, H^{-\infty,-\gamma_1}(B)) = \omega_2\mathcal{H}^{-\infty,-\gamma_2}(\mathbb{R}_+, H^{-\infty,-\gamma_1}(B)) \\ + (1-\omega_2)\mathcal{S}'(\mathbb{R}, H^{-\infty,-\gamma_1}(B))|_{\mathbb{R}_+} \tag{1.33}$$

with the anti-dual of

$$\mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B)).$$

In some considerations such as in the $k = 2$-analogue of Lemma 2.2 below, it makes sense to employ

$$\mathcal{H}^{s,\gamma_2;e}(\mathbb{R}_+, H^{s,\gamma_1}(B)) := \omega_2\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+, H^{s,\gamma_1}(B)) \\ + (1-\omega r_2)r_2^{-e}H^s(\mathbb{R}, H^{s,\gamma}(B))|_{\mathbb{R}_+} \tag{1.34}$$

for any $s, e \in \mathbb{R}$. Then for $B \in \mathfrak{M}_1$, $\dim B = b$, where

$$\mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B)) = \varprojlim_{s,e\in\mathbb{R}} \mathcal{H}^{s,\gamma_2;e}(\mathbb{R}_+, H^{s,\gamma_1}(B)) \tag{1.35}$$

Theorem 1.8. *For every* $h(r_2, v_2, \eta_2) \in M^0_{\mathcal{O}_{v_2}}(B, \boldsymbol{g}_B; \mathbb{R}^{q_2}_{r_2\eta_2})$, $\boldsymbol{g}_B := (\gamma_1, \gamma_1, (-(\vartheta_1 + 1), 0])$ *the operator*

$$\mathrm{Op}^{\gamma_2 - b/2}_{M_{r_2}}(h)(\eta_2) : \mathcal{K}^{0;\gamma_1,\gamma_2}(B^\wedge) \to \mathcal{K}^{0;\gamma_1,\gamma_2}(B^\wedge)$$

for $b := \dim B$ *is continuous.*

Proof. The claimed continuity can be reduced to a Calderón-Vaillancourt argument which is valid also for arbitrary orders k. $\square$

For every $\mu, \gamma_1, \gamma_2 \in \mathbb{R}$ we choose an element $f^\mu(r_2, v_2, \eta_2) \in M^\mu_{\mathcal{O}_{v_2}}(B, \boldsymbol{g}_B; \mathbb{R}^w_\iota \times \mathbb{R}^{q_2}_{r_2\eta_2})$ such that for any sufficiently large $|\iota|$ and $|\eta_2|$ the operators

$$r_2^{-\mu}\mathrm{Op}_M^{\gamma_2-b/2}(f^\mu)(\eta_2) : \mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B)) \to \mathcal{S}^{\gamma_2-\mu}(\mathbb{R}_+, H^{\infty,\beta-\mu}(B)) \tag{1.36}$$

as well as

$$r_2^{-\mu}\mathrm{Op}_M^{\gamma_2-b/2}(f^\mu)(\eta_2) : (\mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B))' \to (\mathcal{S}^{\gamma_2-\mu}(\mathbb{R}_+, H^{\infty,\gamma_1-\mu}(B))' \tag{1.37}$$

are isomorphisms.

For the construction of f^μ we can proceed as follows. We consider the space of holomorphic Mellin symbols

$$M^\mu_{\mathcal{O}_{v_2}}(B, \boldsymbol{g}_B; \mathbb{R}^w_\iota \times \mathbb{R}^{q_2}_{\tilde{\eta}_2}) \tag{1.38}$$

with an extra parameter $\iota \in \mathbb{R}^w$. We obtain elements of the space (1.38) by kernel cut-off from the parameter-dependent space

$$L^\mu(B, \boldsymbol{g}_B; \mathbb{R}^w_\iota \times \Gamma_\lambda \times \mathbb{R}^{q_2}_{\tilde{\eta}_2}) \tag{1.39}$$

Then we can start with a parameter-dependent elliptic $\tilde{l}(\iota, \rho_2, r_2\eta_2)$, $\rho_2 \in \Gamma_\lambda$ in (1.39) and pass via kernel cut-off, where ρ_2 turns to the complex variable v_2 to an element $f^\mu(r_2, \iota, v_2, \eta_2) := \tilde{f}^\mu(\iota, v_2, r_2\eta_2)$ in (1.38). For sufficiently large $|\iota|$, $|\eta_2|$ we can first prove the injectivity of

$$\big(r_2^\mu \mathrm{Op}_M^{\gamma_2-\mu-b/2}((f^\mu)^{-1})(\eta_2)\big) r_2^{-\mu}\mathrm{Op}_M^{\gamma_2-b/2}(f^\mu)(\eta_2)$$

in $\mathcal{K}^{0,\gamma}(B^\wedge)$ which entails the injectivity on $\mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B)))$ and then, via the sesquilinear pairing (1.31) the injectivity also on $(\mathcal{S}^{\gamma_2}(\mathbb{R}_+, H^{\infty,\gamma_1}(B)))'$.

Choose an $f^{-s}(r_2, v_2, \eta_2) \in M^{-s}_{\mathcal{O}_{v_2}}(B, \boldsymbol{g}_B; \mathbb{R}^{q_2}_{r_2\eta_2})$ of this kind and denote the isomorphism (1.37) by

$$R^{-s} = r_2^s \mathrm{Op}_M^{\gamma_k+s-b_1/2}(f^{-s})(\eta_2).$$

For $\gamma = (\gamma_1, \gamma_2)$, $s \in \mathbb{R}$ we define

$$\mathcal{K}^{s,\gamma}(B^\wedge) := R^{-s}\Big(\mathcal{K}^{0,\gamma-s}(B^\wedge)\Big), \tag{1.40}$$

endowed with the norm

$$\|u\|_{\mathcal{K}^{s,\gamma}(B^\wedge)} := \|v\|_{\mathcal{K}^{0,\gamma-s}(B^\wedge)}$$

for $u = R^{-s}v$.

Theorem 1.9. *For every $\mu \in \mathbb{R}, \gamma := (\beta, \gamma_2) \in \mathbb{R}^2$ there exists an element*

$$f^\mu(r_2, v_2, \eta_2) \in M_{\mathcal{O}}^\mu(B, \boldsymbol{g}_B; \mathbb{R}^q_{r_2\eta_2}) \tag{1.41}$$

for $\boldsymbol{g}_B := (\gamma_1, \gamma_1 - \mu, (-(\vartheta_1 + 1), 0])$, such that for any $\eta_2 \neq 0$ the operator

$$r_2^{-\mu}\mathrm{Op}_M^{\gamma_2 - b_1/2}(f^\mu)(\eta_2) : \mathcal{K}^{s,\gamma}(B^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(B^\wedge) \tag{1.42}$$

is an isomorphism for every $s \in \mathbb{R}$. Moreover, an operator (1.42) *for any f^μ as in* (1.41) *is continuous for $\eta_2 \neq 0$.*

Proof. First we write

$$\mathcal{K}^{s,\gamma}(B^\wedge) = R^{-s}\mathcal{K}^{0,\gamma-s}(B^\wedge), \quad \mathcal{K}^{s-\mu,\gamma-\mu}(B^\wedge) = R^{-(s-\mu)}\mathcal{K}^{0,\gamma-(s-\mu)}(B^\wedge).$$

Then, denoting the operator (1.41) to be established by A we can consider $B := R^{s-\mu}AR^{-s} : \mathcal{K}^{0,\gamma-s}(B^\wedge) \to \mathcal{K}^{0,\gamma-(s-\mu)}(B^\wedge)$. It suffices to construct any isomorphism $B : \mathcal{K}^{0,\gamma-s}(B^\wedge) \to \mathcal{K}^{0,\gamma-(s-\mu)}(B^\wedge)$ in our operator class and then return to $A := R^{-(s-\mu)}BR^s$. Such a B can be easily found by the former methods, and then we find the desired A. Then we also obtain the second assertion. □

1.4. Singular Functions and Discrete Asymptotics

For understanding the edge algebra it is essential to look at ideals of operators which are related to singular functions of asymptotics when we approach singularities. In this subsection we establish an approach which is easier than the one developed, e.g., [56]. We focus on constant discrete asymptotics. For brevity we do not consider continuous asymptotics. Let us first look at the traditional edge algebra on a manifold B with edge locally near $s_1(B)$ modeled on $\mathbb{R}^q \times X^\Delta$ for a compact $X \in \mathfrak{M}_0$. Then a discrete asymptotic types associated with weight data (β, Θ) and dimension $n := \dim X$, for a weight $\beta \in \mathbb{R}$ and a weight interval $\Theta = (-(\vartheta + 1), 0]$ for a $\vartheta \in \mathbb{N} \cup \{\infty\}$ is defined as a sequence

$$\mathcal{P} := \{(p_j, m_j)\}_{j=1,\dots,N} \subset \mathbb{C} \times \mathbb{N} \tag{1.43}$$

such that $\pi_{\mathbb{C}}\mathcal{P} := \{p_j\}_{j=1,\dots,N}$ is finite if ϑ is finite, otherwise we assume $\mathrm{Re}\, p_j \to \infty$ for $j \to \infty$ and

$$\pi_{\mathbb{C}}\mathcal{P} \subset \{v \in \mathbb{C} : (n+1)/2 - \beta + \vartheta < \mathrm{Re}\, v < (n+1)/2 - \beta)\}.$$

The space of singular functions $\mathcal{E}_{\mathcal{P}}(X^\wedge)$ over $X^\wedge$ for a finite asymptotic type $\mathcal{P}$ associated with the weight data (γ, Θ) is defined as the set

$$\mathcal{E}_{\mathcal{P}}(X^\wedge) := \{\omega \sum_{j=0}^{N} \sum_{l=0}^{m_j} c_{jl} r^{-p_j} \log^l r : c_{jl} \in C^\infty(X)\}, \tag{1.44}$$

where $\omega(r)$ is a cut-off function. Then we set $\mathcal{K}_\Theta^{s,\beta}(X^\wedge) := \varprojlim_{\varepsilon > 0} \mathcal{K}^{s,\beta+\vartheta-\varepsilon}(X^\wedge)$ and the non-direct sum

$$\mathcal{K}_{\mathcal{P}}^{s,\gamma_1}(X^\wedge) := \mathcal{K}_\Theta^{s,\gamma_1}(X^\wedge) + \mathcal{E}_{\mathcal{P}}(X^\wedge). \tag{1.45}$$

Globally on a manifold $B \in \mathfrak{M}_1$ with edge Y and $\beta \in \mathbb{R}$ we set

$$H^{s,\gamma_1}_{\mathcal{P}}(B) := \omega_1 \mathcal{W}^s(Y, \mathcal{K}^{s,\gamma_1}_{\mathcal{P}}(X^\wedge)) + (1-\omega_1)H^s(2\mathbb{B}), \tag{1.46}$$

cf. also notation (1.26). In addition we set $\mathcal{K}^{s,\gamma_1;e}(X^\wedge) := [r]^{-e}\mathcal{K}^{s,\gamma_1}(X^\wedge)$ for any $e \in \mathbb{R}$ and $\mathcal{K}^{s,\gamma_1;e}_{\mathcal{P}}(X^\wedge) := [r]^{-e}\mathcal{K}^{s,\gamma_1;e}_{\mathcal{P}}(X^\wedge)$. An element C belongs to $L^{-\infty}(B, \boldsymbol{g}_B)$ for a manifold $B \in \mathfrak{M}_1$ with edge and $\boldsymbol{g}_B := (\gamma_1, \gamma_1 - \mu, (0, -(\vartheta+1])$ if it induces continuous maps

$$C : H^{s,\gamma_1}(B) \to H^{\infty,\gamma_1-\mu}_{\mathcal{P}}(B), \quad C^* : H^{s,-\gamma_1+\mu}(B) \to H^{\infty,-\gamma_1}_{\mathcal{S}}(B)$$

for discrete asymptotic types $\mathcal{P}$, $\mathcal{S}$ depending on C. We set $L^{-\infty}(B, \boldsymbol{g}_B; \mathbb{R}^d_\zeta) := \mathcal{S}(\mathbb{R}^d_\zeta, L^{-\infty}(B, \boldsymbol{g}_B))$.

Definition 1.10. An element $g(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma_1}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge))$ is called a Green symbol if

$$g(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma_1;e}(X^\wedge), \mathcal{K}^{\infty,\gamma_1-\mu;\infty}_{\mathcal{P}}(X^\wedge)), \tag{1.47}$$

$$g^*(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,-\gamma_1+\mu;e}(X^\wedge), \mathcal{K}^{\infty,-\gamma_1;\infty}_{\mathcal{S}}(X^\wedge)) \tag{1.48}$$

for all $s, e \in \mathbb{R}$, where "g^*" indicates the $\mathcal{K}^{0,0}(X^\wedge)$-adjoint of g, and $\mathcal{P}$, $\mathcal{S}$ are asymptotic types depending on g.

The symbols of Definition 1.10 generate the class $L^\mu_{\mathrm{G}}(B, \boldsymbol{g}; \mathbb{R}^d) \subset L^\mu(B, \boldsymbol{g}_B; \mathbb{R}^d)$ of Green operators of the edge calculus.

Let us now recall other subclasses of edge operators, namely, smoothing Mellin plus Green operators with constant discrete asymptotics. Let us first give a definition of asymptotic types for smoothing Mellin symbols. Such an asymptotic type is a sequence

$$\mathcal{R} := \{(r_j, n_j)\}_{j\in\mathbb{J}} \subseteq \mathbb{C} \times \mathbb{N}$$

for an index set $\mathbb{J} \subseteq \mathbb{Z}$ such that $\pi_{\mathbb{C}}\mathcal{R} := \{(r_j)\}_{j\in\mathbb{J}}$ intersects every strip $\{v \in \mathbb{C} : c \le \operatorname{Re} v \le c'\}$ for finite $c \le c'$ in a finite set of points.

Definition 1.11. By $M^{-\infty}_{\mathcal{R}}(X)$ we denote the set of all

$$f \in \mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}}\mathcal{R}, L^{-\infty}(X))$$

such that for any $\pi_{\mathbb{C}}\mathcal{R}$-excision function χ we have $\chi f \in L^{-\infty}(X; \Gamma_\lambda))$ for every real λ, uniformly in compact λ-intervals and which are meromorphic with poles at the points of $\pi_{\mathbb{C}}\mathcal{R}$ of multiplicity $n_j + 1$ and Laurent coefficients of finite rank belonging to $L^{-\infty}(X)$.

Smoothing Mellin symbols associated with the weight interval $(-(\vartheta+1), 0]$ for $\vartheta \in \mathbb{N}$ are of the the form

$$m(y,\eta) := r^{-\mu}\omega_{1,\eta} \sum_{j=0}^{\vartheta} \sum_{|\alpha|\le j} r^j \operatorname{Op}^{\gamma_{j\alpha}-n/2}_M(f_{j\alpha})(y)\eta^\alpha \omega'_{1,\eta} \tag{1.49}$$

for $\omega_{1,\eta}(r) = \omega_1(r[\eta])$ with ω_1 being a cut-off function in r, arbitrary

$$f_{j\alpha}(y,v) \in C^\infty(\Omega, M^{-\infty}_{\mathcal{R}_{j\alpha}}(X))$$

for Mellin asymptotic types $\mathcal{R}_{j\alpha}$ and weights

$$\gamma - j \leq \gamma_{j\alpha} \leq \gamma \quad \text{such that} \quad \mathcal{R}_{j\alpha} \cap \Gamma_{(n+1)/2} - \gamma_{j\alpha} = \emptyset,\ n = \dim \mathrm{X}.$$

Applying ideas of [26], we can define singular functions also in another manner. It may be convenient to avoid cut-off functions in the definition of (1.44), or in higher order analogues. For instance, if we replace $\omega(r)$ by a function of the form $e^{-r^{2N}}$ for some $N \in \mathbb{N}$, $N \geq 1$, then we obtain a modified space $\mathcal{E}_{\mathcal{P}}(X^\wedge)$, where (1.45) remains the same, except that the resulting asymptotic type $\mathcal{P}$ satisfies the shadow condition, i.e., $(p,m) \in \mathcal{P}$ entails $(p-j,m) \in \mathcal{P}$ for $j \in \mathbb{N}$ such that $\mathrm{Re}\,(p-j) > (n+1)/2 - \gamma + \vartheta$.

Similar constructions make sense over any $B \in \mathfrak{M}_1$ rather than $X \in \mathfrak{M}_0$ By $L^\mu_{\mathrm{M+G}}(B, \boldsymbol{g}_B) \subset L^\mu(B, \boldsymbol{g}_B)$ we denote the space of all operators $M+G$ generated by symbols $(m+g)(y_2,\eta_2)$ for analogues of m of analogues form as (1.49) and Green symbols g. In a similar manner we can introduce parameter-depending classes with parameter $\zeta \in \mathbb{R}^d$ by adding ζ as an extra covariable together with η. Details for arbitrary $B \in \mathfrak{M}_k$ for any $k \in \mathbb{N}$ are developed in Subsection 2.4 below. The smoothing Mellin symbols, say, for $k=2$ of the class $M^{-\infty}_{\mathcal{R}}(B, \boldsymbol{g}_B) \ni f(v_2)$ are a counterpart of holomorphic Mellin symbols $h(r_2,v_2,\eta_2) \in M^\mu_{\mathcal{O}}(B, \boldsymbol{g}_B; \mathbb{R}^q_{r_2\eta_2})$. Any such Mellin symbol generates a sequence of conormal symbols. The leading one is $h_0(v_2) = h(0,v_2,0)$ which is holomorphic as well.

1.5. Parameter-dependent Edge Calculus

In this subsection we summarize some constructions on parameter-dependent operators on a space $B \in \mathfrak{M}_1$ with edge $Y_1 := s_1(B)$ for $\dim Y_1 = q_1 > 0$. Because of the higher corner calculus below from now on we write (y_1,η_1) rather than $(y,\eta)$and Θ_1 instead of Θ, etc.. We assume that B is locally near Y_1 modeled on $X^\Delta \times \mathbb{R}^{q_1}$ for some compact $X \in \mathfrak{M}_0$. These operators furnish a space

$$L^\mu(B, \boldsymbol{g}_B; \mathbb{R}^d_\zeta) \subseteq L^\mu_{\mathrm{cl}}(s_0(B); \mathbb{R}^d_\zeta) \tag{1.50}$$

for weight data $\boldsymbol{g} = (\gamma_1, \gamma_1 - \mu, \Theta_1)$ with weights $\gamma_1, \gamma_1 - \mu \in \mathbb{R}$ and a weight interval $\Theta_1 = (-(\vartheta_1+1), 0]$ for some $\vartheta_1 \in \mathbb{N}$. The elements of (1.50) consist of sums

$$A(\zeta) = H(\zeta) + M(\zeta) + G(\zeta) + A_{\mathrm{int}}(\zeta) + C(\zeta) \tag{1.51}$$

where $H(\zeta)$ is locally near Y_1 of the form

$$H(\zeta) := \mathrm{Op}_{y_1}\{\omega_1 r_1^{-\mu} \mathrm{Op}_M^{\gamma_1 - n/2}(h)(y_1,\eta_1,\zeta)\omega_1'\} \tag{1.52}$$

for $h(r_1,y_1,v_1,\eta_1,\zeta) := \tilde{h}(r_1,y_1,v_1,r_1\tilde{\eta}_1,r_1\tilde{\zeta})$ for

$$\tilde{h}(r_1,y_1,v_1,\tilde{\eta}_1,\tilde{\zeta}_1) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M^\mu_{\mathcal{O}}(X; \mathbb{R}^{q_k+d}_{\tilde{\eta}_1,\tilde{\zeta}})), \tag{1.53}$$

cut-off functions $\omega_1(r_1) \prec \omega_1'(r_1)$, $M + G \in L^{\mu}_{\mathrm{M+G}}(B, \boldsymbol{g}_B; \mathbb{R}^d_\zeta)$, and $A_{\mathrm{int}} \in (1 - \omega_1)L^{\mu}_{\mathrm{cl}}(s_0(B); \mathbb{R}^d_\zeta)(1 - \omega_1'')$ for cut-off functions $\omega_1(r_1) \succ \omega_1''(r_1)$.

Let

$$a(y_1, \eta_1, \zeta) = \omega \mathrm{Op}_M^{\beta - n/2}(h)(y_1, \eta_1, \zeta)\omega'$$

where the operators of multiplication by $\omega = \omega(r_1)$, $\omega' = \omega'(r_1)$ belong to the interpretation of $a(y_1, \eta_1, \zeta)$ as operator-valued symbol

$$a(y_1, \eta_1, \zeta) \in S^{\mu}(\Omega_1 \times \mathbb{R}^{q_1+d}_{\eta_1,\zeta}; \mathcal{K}^{s,\gamma_1}(X^\wedge), \mathcal{K}^{s-\mu,\gamma_1-\mu}(X^\wedge)). \tag{1.54}$$

Moreveocer, $C(\zeta) \in L^{-\infty}(B, \boldsymbol{g}_B; \mathbb{R}^d_\zeta)$. The full amplitude functions $(a + m + g)(y_1, \eta_1, \zeta)$ of elements $(A + M + G + C)(\zeta) \in L^{\mu}(B, \boldsymbol{g}_B; \mathbb{R}^d_\zeta)$ furnish a subspace of $S^{\mu}(\Omega_1 \times \mathbb{R}^{q_1+d}_{\eta_1,\zeta_1}; \mathcal{K}^{s,\gamma_1}(X^\wedge), \mathcal{K}^{s-\mu,\beta-\mu}(X^\wedge))$ that we denote by

$$R^{\mu}(\mathbb{R}^{q_1} \times \mathbb{R}^{q_1+d}, \boldsymbol{g}_B). \tag{1.55}$$

2. The Higher Iterative Calculus

2.1. Mellin Operators of Arbitrary Order

We now formulate an iterative process of generating higher corner operators. The operators on $M \in \mathfrak{M}_k$ locally near $Y_k := s_k(M)$ modeled on $B^\Delta_{k-1} \times \mathbb{R}^{q_k}$, for some $B := B_{k-1} \in \mathfrak{M}_{k-1}$ of dimension b and $q_k = \dim Y_k$, have the form

$$A = H + M + G + A_{\mathrm{int}} + C \tag{2.1}$$

belonging to a space

$$L^{\mu}(M, \boldsymbol{g}) \subseteq L^{\mu}_{\mathrm{cl}}(s_0(M)) \tag{2.2}$$

for a tuple of weight data $\boldsymbol{g} := (\boldsymbol{g}_i)_{i=1,\dots,k}$, $\boldsymbol{g}_i = (\gamma_i, \gamma_i - \mu, \Theta_i)$, $\Theta_i := (-(\vartheta_i + 1), 0]$, $\vartheta_i \in \mathbb{N}$, where H is locally near $s_k(M)$ under the assumption $\dim s_k(M) > 0$ of the form

$$H := \mathrm{Op}_{y_k}\{\omega_k r_k^{-\mu} \mathrm{Op}^{\gamma_k - b/2}_{M_{r_k}}(h)(y_k, \eta_k)\omega_k'\} \tag{2.3}$$

for $h(r_k, y_k, v_k, \eta_k) := \tilde{h}(r_k, y_k, v_k, r_k\eta_k)$

$$\tilde{h}(r_k, y_k, v_k, \tilde{\eta}_k) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^{q_k}, M^{\mu}_{\mathcal{O}_{v_k}}(B, \boldsymbol{g}_B; \mathbb{R}^{q_k}_{\tilde{\eta}_k})), \tag{2.4}$$

for $\boldsymbol{g}_B := (\boldsymbol{g}_i)_{i=1,\dots,k-1}$, and cut-off functions $\omega_k = \omega_k(r_k), \omega_k' = \omega_k'(r_k)$ with $\omega_k \prec \omega_k'$. The space $M^{\mu}_{\mathcal{O}}(B, \boldsymbol{g}_B; \tilde{\eta}_k)$ is defined as the set of all

$$\tilde{h}(v_k, \tilde{\eta}_k) \in \mathcal{A}(\mathbb{C}_{v_k}, L^{\mu}(B, \boldsymbol{g}_B; \mathbb{R}^{q_k}_{\tilde{\eta}_k})),$$

such that

$$\tilde{h}(\lambda + i\rho, \tilde{\eta}_k) \in L^{\mu}(B, \boldsymbol{g}_B; \Gamma_\lambda \times \mathbb{R}^{q_k}_{\tilde{\eta}_k})$$

for every real λ, uniformly in compact λ-intervals. Here, $v_k = \lambda + i\rho$ and $(\lambda, \tilde{\eta}) \in \mathbb{R}^{1+q_k}$ is the parameter in the edge operator class over B with “sleeping” parameters that are contained in the symbol classes which are of nature as extra edge covariables, cf. the parameter ζ in (1.50).

Moreover, we define $M+G \in L^{\mu}_{\mathrm{M+G}}(M,\boldsymbol{g})$ in a separate step, and employ $A_{\mathrm{int}} \in (1-\omega)L^{\mu}(M \setminus s_k(M), \boldsymbol{g}_B)(1-\omega'')$ for cut-off functions $\omega(r) \succ \omega''(r)$, known from the iterative steps before.

Because of relation (2.2) the operators $A \in L^{\mu}(M,\boldsymbol{g})$ have a standard homogeneous principal symbol $\sigma_0(A)$. Moreover, we define

$$\sigma_k(A)(y_k,\eta_k) := \sigma_k(H)(y_k,\eta_k) + \sigma_k(M+G)(y_k,\eta_k). \tag{2.5}$$

The second term on the right-hand side is defined in Subsection 2.4 below. Concerning H we set

$$\sigma_k(H)(y_k,\eta_k) := r_k^{-\mu}\mathrm{Op}_{M_{r_k}}^{\gamma_k-b/2}(h_0)(y_k,\eta_k) \tag{2.6}$$

for $\eta_k \neq 0$,

$$h_0(r_k,y_k,v_k,\eta_k) := \tilde{h}(0,y_k,v_k,r_k\eta_k).$$

We shall see below that $\sigma_k(A)(y_k,\eta_k)$ defines a family of continuous operators

$$\sigma_k(A)(y_k,\eta_k) := \mathcal{K}^{s,\gamma}(B^{\wedge}) \to \mathcal{K}^{s-\mu,\gamma-\mu}(B^{\wedge}) \tag{2.7}$$

for every $s \in \mathbb{R}$ and $\gamma = (\beta,\gamma_k)$ for $\beta := (\gamma_1,\dots,\gamma_{k-1})$. Since operators $A \in L^{\mu}(M,\boldsymbol{g})$ belong at the same time to a similar operator space over $M \setminus s_k(M) \in \mathfrak{M}_{k-1}$, the definition (2.6) of symbols can be iterated, which gives us first $\sigma_{k-1}(A)(y_{k-1},\eta_{k-1}) : \mathcal{K}^{s,\beta}(C^{\wedge}) \to \mathcal{K}^{s-\mu,\beta-\mu}(C^{\wedge})$ where B locally near $s_{k-1}(B)$ modeled on $C^{\Delta} \times \mathbb{R}^{q_{k-1}}$ for a $C \in \mathfrak{M}_{k-2}$, $\eta_{k-1} \neq 0$, etc., up to $\sigma_0(A)(y_0,\eta_0)$, mentioned before.

Definition 2.1. By $R^{\mu}(\Omega_k \times \mathbb{R}^{q_k},\boldsymbol{g})$ we denote the set of all operator families

$$a(y_k,\eta_k) := \omega_k r_k^{-\mu}\mathrm{Op}_{M_{r_k}}^{\gamma_k-b/2}(h)(y_k,\eta_k)\omega_k' + (m+g)(y_k,\eta_k) \tag{2.8}$$

for arbitrary cut-off functions $\omega_k := \omega_k(r_k)$, $\omega_k' := omega_k'(r_k)$, and

$$h(r_k,y_k,v_k,\eta_k) \in C^{\infty}(\overline{\mathbb{R}}_+ \times \Omega_k, M^{\mu}_{\mathcal{O}_{v_k}}(B,\boldsymbol{g}_B;\mathbb{R}^{q_k}_{r_k\eta_k})).$$

The elements of $R^{\mu}(\Omega_k \times \mathbb{R}^{q_k},\boldsymbol{g})$ are local amplitude functions for the calculus of operators $A \in L^{\mu}(M,\boldsymbol{g})$ close to $s_k(M)$, up to the interior contributions A_{int}. Similarly we have local amplitude functions close to every $s_j(M)$, up to the interior contributions, for every $0 < j < k$. Writing

$$\mathrm{symb}L^{\mu}(M,\boldsymbol{g}) := \{\sigma(A) = (\sigma_j(A))_{j=0,\dots,k} : A \in L^{\mu}(M,\boldsymbol{g})\}$$

we have a principal symbolic map $A \mapsto \sigma(A)$, namely,

$$\sigma : L^{\mu}(M,\boldsymbol{g}) \to \mathrm{symb}(L^{\mu}(M,\boldsymbol{g}).$$

We form the spaces

$$H^{s,\beta}(B) := \omega_{k-1}\mathcal{W}^s(Y_{k-1},\mathcal{K}^{s,\beta}(C^{\Delta})) + (1-\omega_{k-1}))H^{s,\delta}(2\mathbb{B}) \tag{2.9}$$

for $\delta := (\gamma_1,\dots,\gamma_{k-2})$ and a cut-off function ω_{k-1} in r_{k-1}.

Using the space $H^{\infty,\beta}(B) = \bigcap_{s\in\mathbb{R}} H^{s,\beta}(B)$ for compact $B \in \mathfrak{M}_{k-1}$ and any weight tuple $\beta = (\gamma_1, \dots, \gamma_{k-1})$ we set

(2.10)
$$\mathcal{S}^{\gamma_k}(\mathbb{R}_+, H^{\infty,\beta}(B)) := \omega_k \mathcal{H}^{\infty,\gamma_k}(\mathbb{R}_+, H^{\infty,\beta}(B)) + (1-\omega_k)\mathcal{S}(\mathbb{R}, H^{\infty,\beta}(B))|_{\mathbb{R}_+},$$

where ω_k is a cut-off function on the r_k half-axis. Since the Kegel spaces involved in (2.7) are studied later on instead of (2.7) we interpret the symbol $\sigma_k(A)$ according to

Lemma 2.2. *The symbol $\sigma_k(A)(y_k, \eta_k)$ induces a family of continuous maps*

$$\sigma_k(A)(y_k, \eta_k) := \mathcal{S}^{\gamma_k}(\mathbb{R}_+, H^{\infty,\beta}(B)) \to \mathcal{S}^{\gamma_k-\mu}(\mathbb{R}_+, H^{\infty,\beta-\mu}(B)). \tag{2.11}$$

Proof. We focus on the Mellin part of A. The M plus G part is simple and left to the reader. Let $Mu(r) := r^{-\mu}\mathrm{Op}_M^{\gamma}(f)$, first for a Mellin symbol $f(v) \in M_{\mathcal{O}}^{\mu}(H, \widetilde{H})$ with constant coefficients in r, taking values in a $\mathcal{L}(H, \tilde{H})$ for Hilbert spaces H, $\tilde{H}$. We show that

$$M : \mathcal{S}^{\gamma}(\mathbb{R}_+, H) \to \mathcal{S}^{\gamma}(\mathbb{R}_+, H)$$

is continuous. Here

$$\mathcal{S}^{\gamma}(\mathbb{R}_+, H) := \omega(r)\mathcal{H}^{\infty,\gamma}(\mathbb{R}_+, H) + (1-\omega(r))\mathcal{S}(\mathbb{R}, H)|_{\mathbb{R}_+}$$

for some cut-off function ω. Let us write $M = \big(\omega + (1-\omega)\big)M\big(\tilde{\omega} + (1-\tilde{\omega})\big)$. Then the desired mapping property for $\omega M \tilde{\omega}$ and $\omega M(1-\tilde{\omega})$ is certainly true. For $(1-\omega)M\tilde{\omega}$ and $(1-\omega)M(1-\tilde{\omega})$ we may argue in terms of commutation relations, e.g., $(1-\omega)M\tilde{\omega}r^{-N} = r^{-N}(1-\omega)M_N\tilde{\omega}$ for any $N \in \mathbb{N}$ and a Mellin operator M_N with a translated Mellin symbol. For derivaties in r we combine commutation relations with $r\partial r$ with powers of r treated before. Similar arguments hold when we replace H, $\widetilde{H}$ by projective limits of Hilbert spaces. □

In order to understand more on the iterative structure of edge symbols we consider as an example an edge-degenerate differential operator

$$A = r_2^{-\mu} \sum_{j_2+|\alpha_2|\le\mu} a_{j_2,\alpha_2}(-r_2\partial_{r_2})^{j_2}(r_2 D_{y_2})^{\alpha_2} \tag{2.12}$$

for coefficients $a_{j_2,\alpha_2}(r_2, y_2) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^{q_2}, \mathrm{Diff}_{\deg}^{\mu-(j_2+|\alpha_2|)}(B))$ for a $B \in \mathfrak{M}_1$ locally close to $s_1(B)$ modeled on $X^\Delta \times \Omega_1$ for an $X \in \mathfrak{M}_0$ and insert once again

$$a_{j_2,\alpha_2} = r_1^{-\mu+(j_2+|\alpha_2|)} \sum_{j_1+|\alpha_1|\le\mu-(j_2+|\alpha_2|)} b_{j_1,\alpha_1}(-r_1\partial_{r_1})^{j_1}(r_1 D_{y_1})^{\alpha_1} \tag{2.13}$$

for coefficients $b_{j_1,\alpha_1}(r_1, y_1) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^{q_1}, \mathrm{Diff}^{\mu-(j_1+|\alpha_1|)}(X))$. Then we obtain a corner differential operator

(2.14)
$$A = r_2^{-\mu} r_1^{-\mu+(j_2+|\alpha_2|)} \sum_{j_2+|\alpha_2|\le\mu}\ \sum_{j_1+|\alpha_1|\le\mu-(j_2+|\alpha_2|)} b_{j_1,\alpha_1}(-r_1\partial_{r_1})^{j_1}(r_1 D_{y_1})^{\alpha_1} \\ (-r_2\partial_{r_2})^{j_2}(r_2 D_{y_2})^{\alpha_2}$$

which is the same as

$$A = r_2^{-\mu} r_1^{-\mu} \sum_{j_2+|\alpha_2|\le\mu} \sum_{j_1+|\alpha_1|\le\mu-(j_2+|\alpha_2|)} b_{j_1,\alpha_1}(-r_1\partial_{r_1})^{j_1}(r_1 D_{y_1})^{\alpha_1} (-r_1 r_2 \partial_{r_2})^{j_2}(r_1 r_2 D_{y_2})^{\alpha_2}. \tag{2.15}$$

Off $r_2 = 0$ the operator C is edge-degenerate for the edge $\Omega_1 \times \mathbb{R}_{+,r_2} \times \Omega_2$ and has as such an edge symbol

$$\sigma_1(A)(y_1, r_2, y_2, \eta_1, \rho_2, \eta_2) = r_1^{-\mu} \sum_{j_2+|\alpha_2|\le\mu} \sum_{j_1+|\alpha_1|\le\mu-(j_2+|\alpha_2|)} r_2^{-\mu} b_{j_1,\alpha_1} (-r_1\partial_{r_1})^{j_1}(r_1\eta_1)^{\alpha_1}(r_1 r_2 i\rho_2)^{j_2}(r_1 r_2 \eta_2)^{\alpha_2}. \tag{2.16}$$

In order to avoid confusion compared with notation in (1.3) the role of y_1 now plays the tuple of variables (y_1, r_2, y_2) and of η_1 the covariables (η_1, ρ_2, η_2) which are multiplied by r_1 as it ought to be because of the edge-degenerate behaviour in r_1. However, close to $r_2 = 0$ the operator A is corner-degenerate, and

$$\sigma_2(A)(y_2, \eta_2) = r_2^{-\mu} \mathrm{Op}_{M_{r_2}}^{\gamma_2 - b/2}(h_0)(y_2, \eta_2) \tag{2.17}$$

for

$$h_0(y_2, v_2, \eta_2) = \tilde{h}(0, y_2, v_2, r_2\eta_2) = \sum_{j_2+|\alpha_2|\le\mu} \sum_{j_1+|\alpha_1|\le\mu-(j_2+|\alpha_2|)} b_{j_1,\alpha_1}|_{r_2=0} r_1^{-\mu}(-r_1\partial_{r_1})^{j_1}(r_1 D_{y_1})^{\alpha_1}(r_1 v_2)^{j_2}(r_1 r_2 \eta_2)^{\alpha_2}, \tag{2.18}$$

i.e., $\sigma_2(A)(y_2, \eta_2)$ is described by a Mellin symbol in $C^\infty(\mathbb{R}^{q_2}, M_{\mathcal{O}_{v_2}}^{\mu}(B, \boldsymbol{g}; \mathbb{R}^{q_2}_{r_2\eta_2}))$.

2.2. Compositions

Theorem 2.3. *Let $M \in \mathfrak{M}_k$, $\dim s_k(M) > 0$. Then $P \in L^\mu(M, \boldsymbol{p})$ for $\boldsymbol{p} = (\gamma_i - \nu, \gamma_i - (\mu+\nu), \Theta_i)_{i=1,\dots,k}$ and $Q \in L^\nu(M, \boldsymbol{q})$ for $\boldsymbol{q} = (\gamma_i, \gamma_i - \nu, \Theta_i)_{i=1,\dots,k}$ implies $PQ \in L^{\mu+\nu}(M, \boldsymbol{p} \circ \boldsymbol{q})$ (when one of the factors is properly supported, in obvious meaning) and*

$$\sigma_j(PQ) = \sigma_j(P)\sigma_j(Q), \quad j = 1, \dots, k. \tag{2.19}$$

Proof. Similarly as (2.1) we have

$$P = H + M + G + A_{\mathrm{int}} + C, \quad Q = F + N + L + B_{\mathrm{int}} + D$$

with obvious notation. Let us first show that HF is of the form

$$\mathrm{Op}_{y_k}\{\omega_k r_k^{-(\mu+\nu)} \mathrm{Op}_M^{\gamma_k - b/2}(l)(y_k, \eta_k)\omega_k' + (m+g)(y_k, \eta_k)\} \tag{2.20}$$

for cut-off functions $\omega_k(r_k), \omega_k'(r_k)$, a Mellin symbol

$$l(r_k, y_k, v_k, r_k\eta_k) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^{q_k}, M_{\mathcal{O}}^{\mu+\nu}(B, \boldsymbol{g}_B; \mathbb{R}^{q_k}_{r_k\eta_k})), \tag{2.21}$$

and $(m+g)(y_k, \eta_k) \in R^{\mu+\nu}_{\mathrm{M+G}}(\mathbb{R}^{q_k} \times \mathbb{R}^{q_k}, \boldsymbol{g}_\beta)$. Writing

$$P := \mathrm{Op}_{y_k}\{\sigma_k r_k^{-\mu} \mathrm{Op}_M^{\gamma_k - \nu - b/2}(h)(y_k, \eta_k)\sigma_k'\},$$

and

$$Q := \mathrm{Op}_{y_k}\{\tilde{\omega}_k r_k^{-\nu} \mathrm{Op}_M^{\gamma_k - b/2}(m)(y_k,\eta_k)\tilde{\omega}_k'\}$$

we have

(2.22)

$$\begin{aligned} PQ =& \mathrm{Op}_{y_k}\{\omega_k r_k^{-\mu} \mathrm{Op}_M^{\gamma_k-\nu-b/2}(h)(y_k,\eta_k)\omega_k'\} \mathrm{Op}_{y_k}\{\tilde{\omega}_k r_k^{-\nu} \mathrm{Op}_M^{\gamma_k-b/2}(m)(y_k,\eta_k)\tilde{\omega}_k'\} \\ &= \mathrm{Op}_{y_k}\{\omega_k r_k^{-(\mu+\nu)} \mathrm{Op}_M^{\gamma_k-b/2}(T^{-\nu}h)(y_k,\eta_k)\omega_k'\} \mathrm{Op}_{y_k}\{\tilde{\omega}_k \mathrm{Op}_M^{\gamma_k-b/2}(m)(y_k,\eta_k)\tilde{\omega}_k'\} \\ &= \mathrm{Op}_{y_k}\{\omega_k r_k^{-(\mu+\nu)} \mathrm{Op}_M^{\gamma_k-b/2}(T^{-\nu}h)(y_k,\eta_k)\omega_k'\} \# \{\tilde{\omega}_k \mathrm{Op}_M^{\gamma_k-b/2}(m)(y_k,\eta_k)\tilde{\omega}_k'\} \\ &= \mathrm{Op}_{y_k}\{\omega_k r_k^{-(\mu+\nu)} \mathrm{Op}_M^{\gamma_k-b/2}(T^{-\nu}h)(y_k,\eta_k)\} \# \{\mathrm{Op}_M^{\gamma_k-b/2}(m)(y_k,\eta_k)\tilde{\omega}_k'\} + G \end{aligned}$$

for

$$(2.23) \qquad \begin{aligned} G =& \mathrm{Op}_{y_k}\{\omega_k r_k^{-(\mu+\nu)} \mathrm{Op}_M^{\gamma_k-b/2}(T^{-\nu}h)(y_k,\eta_k)(\omega_k'\tilde{\omega}_{k-1}\} \\ & \#\{\mathrm{Op}_M^{\gamma_k-b/2}(m)(y_k,\eta_k)\tilde{\omega}_k'\}. \end{aligned}$$

The computation of the Fourier-Mellin Leibniz product on the right-hand side of (2.22) applies the rules of oscillatory integrals, using the Mellin-modified version with operator-valued symbols and twisted symbolic estimates, similarly as methods in Seiler [66]. The convergence of the Mellin oscillatory part employs the fact that after finitely often differentiating amplitude functions in r_k or v_k we obtain a decay which implies convergence of integrals over $\mathbb{R}_+ \times \mathbb{R} \ni (r_k, \mathrm{Im}\, v_k)$ when we apply any semi-norm of the involved operator algebras. This ensures the shape of (2.21), including the combination of variables in the form $r_k\eta_k$, the specific arguments for holomorphic dependence of (2.21) on $v_k \in \mathbb{C}$ and the required shape of remainder terms. The Fourier part of oscillatory integrals can be treated in an analogous manner. The expression (2.23) belongs to $L_{\mathrm{G}}^{\mu+\nu}(M, \boldsymbol{p}\circ\boldsymbol{q})$, according to the rules of treating Mellin operator compositions with holomorphic symbols and a factor $\omega-1$ in the middle, for some cut-off function ω. What concerns the M+G-contributions we employ that those form an ideal. The same is true when compositions contain an "int"-factor which preserves "int" when the other factor in P or Q is of that kind, or of Mellin plus Green nature when the other factor belongs to the M+G operator class.

What concerns the composition rule for symbols the only new aspect compared with singularity order $< k$ is relation (2.19) for $j = k$. This follows from limit expressions of the kind

$$\sigma_j(P)(y_k,\eta_k) = \delta^{-\nu} \lim_{\delta\to\infty} \kappa_\delta^{-1} p(y_k, \delta\eta_k)\kappa_\delta$$

when we represent, for instance, the operator P locally near $s_k(M)$ by an amplitude function $p(y_k,\eta_k) \in R^\nu(\Omega_k \times \mathbb{R}^{q_k}, \boldsymbol{g})$, cf. Definition 2.1. □

2.3. Higher Kegel and Wedge Spaces

Recall that notation of weights will be employed as follows. We define weighted spaces of smoothness $s \in \mathbb{R}$ and weight $\gamma := (\gamma_1, \dots, \gamma_k)$ for any $k \geq 2$ and we define

$$\gamma - \mu := (\gamma_1 - \mu, \dots, \gamma_k - \mu)$$

for any $\mu \in \mathbb{R}$. In order to have simple references to constructions for smaller singularity orders, we set

$$\beta := (\gamma_1, \dots, \gamma_{k-1}),\ \delta := (\gamma_1, \dots, \gamma_{k-2})$$

and also $\gamma = (\beta, \gamma_k)$. Kegel spaces are the framework for expressing operator-valued symbols of operators A on $M \in \mathfrak{M}_k$. In fact, the construction of our higher corner operators A refers to local representaions of M close to every $y_j \in s_j(M)$ as

$$B^{\Delta}_{j-1} \times \Omega_j \tag{2.24}$$

for some $B_{j-1} \in \mathfrak{M}_{j-1}$, $j = 1, \dots, k$, successively yields the principal symbolic hierarchy

$$(\sigma_j(A)(y_j, \eta_j))_{j=1,\dots,k},$$

together with the standard homogeneous principal symbol $\sigma_0(A)$ on $s_0(M)$. Then $\sigma_j(A)(y_j, \eta_j))$ is a family of operators between weighted Kegel spaces

$$\sigma_j(A)(y_j, \eta_j)) : \mathcal{K}^{s,\gamma_{(j)}}(B^{\wedge}_{j-1}) \to \mathcal{K}^{s-\mu,\gamma_{(j)}-\mu}(B^{\wedge}_{j-1})$$

for $(y_j, \eta_j) \in \mathbb{R}^{q_j} \times (\mathbb{R}^{q_j} \setminus \{0\})$ all $s \in \mathbb{R}$ and weight tuples $\gamma_{(j)} = (\gamma_1, \dots, \gamma_j) \in \mathbb{R}^j$, where $\gamma_{(j)} - \mu = (\gamma_1 - \mu, \dots, \gamma_j - \mu)$. In this section we introduce and study these higher Kegel spaces. By virtue of the iterative approach we may assume that the cases for $j < k$ are treated. Thus we may focus on the case $j = k$, and we set $B := X_{k-1}$, assuming that B is locally near $s_{k-1}(B) = Y_{k-1}$ identified with $C^{\Delta} \times Y_{k-1}$ for some $C \in \mathfrak{M}_{k-2}$. We use the fact that $2\mathbb{B} \in \mathfrak{M}_{k-2}$.

Definition 2.4. For compact $B \in \mathfrak{M}_{k-1}$ with edge Y_{k-1}, locally near Y_{k-1} modeled on $C^{\Delta} \times Y_{k-1}$ for a $C \in \mathfrak{M}_{k-2}$, $C^{\wedge} = \mathbb{R}_{+,r_{k-1}} \times C$, and for weights $\gamma := (\beta, \gamma_k)$ we define

$$\begin{aligned}\mathcal{K}^{0,\gamma}(B^{\wedge}) := {} & \omega_k \omega_{k-1} \mathcal{H}^{0,\gamma_k}(\mathbb{R}_+ \times Y_{k-1}, \mathcal{K}^{0,\beta}(C^{\wedge})) \\ & + (1-\omega_k)\omega_{k-1} \mathcal{H}^{0,0}(\mathbb{R}_+ \times Y_{k-1}, \mathcal{K}^{0,\beta}(C^{\wedge})) \\ & + (1-\omega_k)(1-\omega_{k-1})\mathcal{K}^{0;\delta,0}((2\mathbb{B})^{\wedge})) + \omega_k(1-\omega_{k-1})\mathcal{K}^{0;\delta,\gamma_k}((2\mathbb{B})^{\wedge}))\end{aligned} \tag{2.25}$$

for cut-off functions $\omega_k = \omega_k(r_k)$, $\omega_{k-1} = \omega_{k-1}(r_{k-1})$.

In Definition 2.4 we employ that the double $2\mathbb{B}$ belongs to $\mathfrak{M}_{k-2}$. The spaces in (2.25) are Hilbert spaces and $\mathcal{K}^{0,\gamma}(B^{\wedge})$ is endowed with the scalar product of the non-direct sum. In particular, we fix the scalar product of $\mathcal{K}^{0,0}(B^{\wedge})$ as a reference scalar product where the second zero has the meaning of multiple weights

$$0 := (0, \dots, 0),\ (k \,\text{times}). \tag{2.26}$$

We have a non-degenerate sesquilinear pairing

$$(\cdot,\cdot)_{\mathcal{K}^{0,0}(B^\wedge)} : \mathcal{K}^{0;\gamma}(B^\wedge) \times \mathcal{K}^{0;-\gamma}(B^\wedge) \to \mathbb{C}$$

for any $\gamma = (\beta, \gamma_k) \in \mathbb{R}^k$, and we have a natural inclusion

$$\mathcal{S}^{\gamma_k}(\mathbb{R}_+, H^{\infty,\beta}(B)) \subseteq \mathcal{K}^{0;\beta,\gamma_k}(B^\wedge). \tag{2.27}$$

This allows us to identify the space

$$\begin{aligned}(\mathcal{S}^{\gamma_k}(\mathbb{R}_+, H^{\infty,\beta}(B)))' = {}& \omega_k \mathcal{H}^{-\infty,-\gamma_k}(\mathbb{R}_+, H^{-\infty,-\beta}(B)) \\ &+ (1-\omega_k)\mathcal{S}'(\mathbb{R}, H^{-\infty,-\beta}(B))|_{\mathbb{R}_+}\end{aligned} \tag{2.28}$$

with the anti-dual of $\mathcal{S}^{\gamma_k}(\mathbb{R}_+, H^{\infty,\beta}(B))$.

We can also form spaces

$$\begin{aligned}\mathcal{H}^{s,\gamma_k;e}(\mathbb{R}_+, H^{s,\beta}(B)) := {}& \omega_k \mathcal{H}^{s,\gamma_k}(\mathbb{R}_+, H^{s,\beta}(B)) \\ &+ (1-\omega_k) r_k^{-e} H^s(\mathbb{R}, H^{s,\beta}(B))|_{\mathbb{R}_+}\end{aligned} \tag{2.29}$$

for any $s, e \in \mathbb{R}$. Then for $B \in \mathfrak{M}_{k-1}$, $\dim B = b$, we have

$$\mathcal{S}^{\gamma_k}(\mathbb{R}_+, H^{\infty,\beta}(B)) = \varprojlim_{s,e\in\mathbb{R}} \mathcal{H}^{s,\gamma_k;e}(\mathbb{R}_+, H^{s,\beta}(B)) \tag{2.30}$$

and

$$\mathcal{H}^{s,\gamma_k;e}(\mathbb{R}_+, H^{s,\beta}(B)) \subseteq \mathcal{K}^{0,0}(B^\wedge)$$

for $s, e \geq 0$, β, $\gamma_k \geq 0$, where $\mathcal{H}^{s,\gamma_k;e}(\mathbb{R}_+, H^{s,\beta}(B))$ can be identified with the anti-dual of $\mathcal{H}^{-s,-\gamma_k;-e}(\mathbb{R}_+, H^{-s,-\beta}(B))$ with respect to the $\mathcal{K}^{0,0}(B^\wedge)$-scalar product and vice versa.

Theorem 2.5. *For every*

$$h(r_k, v_k, \eta_k) \in M^0_{\mathcal{O}_{v_k}}(B, \boldsymbol{g}_B; \mathbb{R}^{q_k}_{r_k\eta_k})$$

for $\boldsymbol{g}_B := (\gamma_j, \gamma_j, (-(\vartheta_j+1), 0])_{j=1,\dots,k-1}$ *the operator*

$$\mathrm{Op}^{\gamma_k-b/2}_{M_{r_k}}(h)(\eta_k) : \mathcal{K}^{0,\gamma}(B^\wedge) \to \mathcal{K}^{0,\gamma}(B^\wedge) \tag{2.31}$$

is continuous.

The proof will be given below.

Theorem 2.6. *Let* $h(r_k, v_k, \eta_k) \in M^\mu_{\mathcal{O}}(B, \boldsymbol{g}_B; \mathbb{R}^d_{r_k\eta_k})$, $\boldsymbol{g}_B := (\gamma_i, \gamma_i-\mu, \Theta_i)_{i=1,\dots,k-1}$; *then*

$$r_k^{-\mu}\mathrm{Op}^{\gamma_k-b/2}_{M_{r_k}}(h)(\eta_k) : \mathcal{S}^{\gamma_k}(\mathbb{R}_+, H^{\infty,\beta}(B)) \to \mathcal{S}^{\gamma_k-\mu}(\mathbb{R}_+, H^{\infty,\beta-\mu}(B)) \tag{2.32}$$

is continuous for every $\beta \in \mathbb{R}^{k-1}$, $\gamma_k \in \mathbb{R}$ *and* $\eta_k \neq 0$.

Proof. The proof follows the lines of Lemma 2.2. □

For every $\mu \in \mathbb{R}$, $\gamma \in \mathbb{R}^k$ and $B \in \mathfrak{M}_{k-1}$ of dimension b we choose an element $f^\mu(r_k, v_k, \eta_k) \in M^\mu_{\mathcal{O}_{v_k}}(B, \boldsymbol{g}_B; \mathbb{R}^w_{\iota_k} \times \mathbb{R}^{q_k}_{r_k\eta_k})$, $\gamma = (\beta, \gamma_k)$, such that the operators

$$r_k^{-\mu}\mathrm{Op}_M^{\gamma_k - b/2}(f^\mu)(\eta_k) : \mathcal{S}^{\gamma_k}(\mathbb{R}_+, H^{\infty,\beta}(B)) \to \mathcal{S}^{\gamma_k-\mu}(\mathbb{R}_+, H^{\infty,\beta-\mu}(B))) \tag{2.33}$$

as well as

$$r_k^{-\mu}\mathrm{Op}_M^{\gamma_k - b/2}(f^\mu)(\eta_k) : (\mathcal{S}^{\gamma_k}(\mathbb{R}_+, H^{\infty,\beta}(B))' \to (\mathcal{S}^{\gamma_k-\mu}(\mathbb{R}_+, H^{\infty,\beta-\mu}(B)))' \tag{2.34}$$

are isomorphisms for any $\eta_k \neq 0$.

Remark 2.7. *The element $f^\mu(r_k, v_k, \eta_k) \in M^\mu_{\mathcal{O}_{v_k}}(B, \boldsymbol{g}_\beta; \mathbb{R}^w_{\iota_k} \times \mathbb{R}^{q_k}_{r_k\eta_k})$ can be chosen in such a way that isomorphisms* (2.33) *and* (2.34) *are induced for all $c \leq \gamma_i \leq c'$ for arbitrary fixed $c \leq c'$. This only requires that the additional parameters $\iota_k \in \mathbb{R}^w$ at every level of singularity are of sufficiently large absolute value. Therefore, if we have some fixed tuples of weights in mind, we can arrange the corresponding isomorphisms for all γ_i and $-\gamma_i$ for $i = 1, \dots, k$ at the same time, using, if necesary, translations and dilations in the complex variables v_j for all j.*

Definition 2.8. Let s, $\gamma_k \in \mathbb{R}$, $\beta \in \mathbb{R}^{k-1}$, and let $B \in \mathfrak{M}_{k-1}$ be of dimension b. Choose an $f^{-s}(r_k, v_k, \eta_k) \in M^{-s}_{\mathcal{O}_{v_k}}(B, \boldsymbol{g}_B; \mathbb{R}^{q_k}_{r_k\eta_k})$ and denote the above-mentioned isomorphism (2.33) or (2.34) by

$$R^{-s} := r_k^s \mathrm{Op}_M^{\gamma_k + s - b/2}(f^{-s})(\eta_k),$$

$\eta_k \neq 0$. We define

$$\mathcal{K}^{s,\gamma}(B^\wedge) := \mathcal{K}^{s;\beta,\gamma_k}(B^\wedge) := R^{-s}\Big(\mathcal{K}^{0;\beta-s,\gamma_k-s}(B^\wedge)\Big), \tag{2.35}$$

for $\gamma := (\beta, \gamma_k)$, $\beta := (\gamma_1, \dots, \gamma_{k-1})$.

Definition 2.9. Let H and $\widetilde{H}$ be Hilbert spaces with group action κ and $\tilde{\kappa}$, respectively. By $S^0(\mathbb{R}^q \times \mathbb{R}^q; H, \widetilde{H})_{\mathrm{cv}}$ we denote the set of all $a(y,\eta) \in C^\infty(\mathbb{R}^{2q}, \mathcal{L}(H, \widetilde{H}))$ such that

$$\pi(a) := \sup\{\|\{\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D_y^\alpha D_\eta^\beta a(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})} : (y,\eta) \in \mathbb{R}^{2q}, \alpha \leq \boldsymbol{\alpha}, \beta \leq \boldsymbol{\beta}\} \tag{2.36}$$

is finite for $\boldsymbol{\alpha} := (M+1, \dots, M+1)$, $\boldsymbol{\beta} := (1, \dots, 1)$, with $M \in \mathbb{N}$ being a constant belonging to the norm growth of $\tilde{\kappa}$ in $\mathcal{L}(\widetilde{H})$. Moreover, let $S^0(\mathbb{R}_+ \times \mathbb{R}^q \times \Gamma_{(b+1)/2-\nu} \times \mathbb{R}^q; H, \widetilde{H})_{\mathrm{CV}}$ denote the set of all $f(r, y, (b+1)/2 - \nu + i\rho) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^q \times \Gamma_{(b+1)/2-\nu} \times \mathbb{R}^q, \mathcal{L}(H, \widetilde{H}))$ such that

$$a(t, y, \rho, \eta) := f(e^{-t}, y, (b+1)/2 - \nu + i\rho, \eta) \in S^0(\mathbb{R} \times \mathbb{R}^q \times \mathbb{R} \times \mathbb{R}^q; H, \widetilde{H})_{\mathrm{cv}}. \tag{2.37}$$

Theorem 2.10. *Let H and $\widetilde{H}$ be Hilbert spaces with group action κ and $\tilde{\kappa}$, respectively, and let $f(r, y, (b+1)/2 - \nu + i\rho, \eta) \in S^0(\mathbb{R}_+ \times \mathbb{R}^q \times \Gamma_{(b+1)/2-\nu} \times \mathbb{R}^q; H, \widetilde{H})_{\mathrm{CV}}$. Then $\mathrm{Op}_y\mathrm{Op}_M^{\nu - b/2}(f)$ induces a continuous operator*

$$\mathrm{Op}_y\mathrm{Op}_M^{\nu - b/2}(f) : \mathcal{H}^{0,\nu}(\mathbb{R}_+ \times \mathbb{R}^q, H) \to \mathcal{H}^{0,\nu}(\mathbb{R}_+ \times \mathbb{R}^q, \widetilde{H}),$$

and we have $\|\mathrm{Op}_y\mathrm{Op}_M^{\nu-b/2}(f)\|_{\mathcal{L}(\mathcal{H}^{0,\nu}(\mathbb{R}_+\times\mathbb{R}^q,H),\mathcal{H}^{0,\nu}(\mathbb{R}_+\times\mathbb{R}^q,\widetilde{H}))} \leq c\pi(f)$ *for a constant* $c>0$ *independent of* f.

Proof. The proof follows the lines of Seiler, [64]. □

Proof of Theorem 2.5. The proof is a consequence of iteratively applying Theorem 2.10. By virtue of Definition 2.4 we can write

$$\mathcal{K}^{0,\gamma}(B^\wedge) = \mathbb{H}_k + \mathbb{L}_k$$

for cut-off functions $\omega_k := \omega_k(r_k)$, $\omega_{k-1} := \omega_{k-1}(r_{k-1})$,

$$\mathbb{H}_k := \omega_k\omega_{k-1}\mathcal{H}^{0,\gamma_k}(\mathbb{R}_+\times Y_{k-1},\mathcal{K}^{0,\beta}(C^\wedge))+(1-\omega_k)\omega_{k-1}\mathcal{H}^{0,0}(\mathbb{R}_+\times Y_{k-1},\mathcal{K}^{0,\beta}(C^\wedge))$$

and

$$\mathbb{L}_k := (1-\omega_k)(1-\omega_{k-1})\mathcal{K}^{0;\delta,0}((2\mathbb{B})^\wedge)) + \omega_k(1-\omega_{k-1})\mathcal{K}^{0;\delta,\gamma_k}((2\mathbb{B})^\wedge)).$$

The claimed continuity in spaces $\mathbb{H}_k$ follows from Theorem 2.10 by applying charts on Y_{k-1} to $\mathbb{R}^{q_{k-1}}$ and a subsequent partition of unity. The spaces contained in $\mathbb{L}_k$ are of smaller singularity order and the continuity holds because of the iteration step before. □

Corollary 2.11. *The spaces* $\mathcal{K}^{s,\gamma}(B^\wedge)$ *are independent of the choice of* R^{-s} *in Definition* 2.8.

In fact, let $\tilde{R}^{-s}$ be an another order reducing isomorphism of analogous kind as R^{-s} and denote the resulting space by $\tilde{\mathcal{K}}^{s,\gamma}(B^\wedge)$. Then

$$R^s\tilde{\mathcal{K}}^{s,\gamma}(B^\wedge) = R^s\tilde{R}^{-s}\mathcal{K}^{0,\gamma-s}(B^\wedge) = \mathcal{K}^{0,\gamma-s}(B^\wedge)$$

since by Theorem 2.5 the operator $R^s\tilde{R}^{-s} : \mathcal{K}^{0,\gamma-s}(B^\wedge) \to \mathcal{K}^{0,\gamma-s}(B^\wedge)$ is an isomorphism. Thus $\mathcal{K}^{s,\gamma}(B^\wedge) = \tilde{\mathcal{K}}^{s,\gamma}(B^\wedge)$.

Remark 2.12. *The operators*

$$A := r_k^{-\mu}\mathrm{Op}_M^{\gamma_k-b/2}(f)(\eta_k) : \mathcal{K}^{s,\gamma}(B^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(B^\wedge) \tag{2.38}$$

for any $f(r_k,v_k,\eta_k) \in M^\mu_{\mathcal{O}_{v_k}}(B,\boldsymbol{g}_B;\mathbb{R}^{q_k}_{r_k\eta_k})$ *are continuous for all* $s\in\mathbb{R}$. *Moreover, we have natural inclusions* $\mathcal{K}^{s',\gamma}(B^\wedge) \hookrightarrow \mathcal{K}^{s,\gamma}(B^\wedge)$ *for* $s' \geq 0$, *since*

$$M^\mu_{\mathcal{O}}(B,\boldsymbol{g}_\beta;\mathbb{R}^{q_k}_{r_k\eta_k}) \subseteq M^{\mu'}_{\mathcal{O}}(B,\boldsymbol{g}_B;\mathbb{R}^{q_k}_{r_k\eta_k})$$

for $\mu' \geq \mu, \boldsymbol{g}'_B = (\gamma_j,\gamma_j-\mu',\Theta_j)$. *Then* (2.38) *can be applied for* $\mu \leq 0$, $\mu' = 0$.

The continuity (2.38) for $s=0$, $\mu=0$ is stated in Theorem 2.5. That means $A = (R^{-s})^{-1}AR^{-s} : \mathcal{K}^{0,\gamma-s}(B^\wedge) \to \mathcal{K}^{0,\gamma}(B^\wedge)$ has the form $\mathrm{Op}_M^{\gamma_k-s-b/2}(f_0)(\eta_k)$ for some $f_0(r_k,v_k,\eta_k) \in M^\mu_{\mathcal{O}_{v_k}}(B,\boldsymbol{g}_B;\mathbb{R}^{q_k}_{r_k\eta_k})$ is continuous. Therefore, using Definition 2.8 we obtain the asserted continuity of A, cf. relations (1.17), (1.18) which are valid for arbitrary k.

For any $e\in\mathbb{R}$ we set

$$\mathcal{K}^{s,\gamma;e}(B^\wedge) := [r_k]^{-e}\mathcal{K}^{s,\gamma}(B^\wedge)$$

or, equivalently,

$$\mathcal{K}^{s,\gamma;e}(B^\wedge) = r_k^{-e}\mathcal{K}^{s;\beta,\gamma_k+e}(B^\wedge). \tag{2.39}$$

Remark 2.13. *We have*

$$\mathcal{K}^{s,\gamma;e}(B^\wedge) = R^{-s}\mathcal{K}^{0,\gamma-s;e}(B^\wedge).$$

In fact, since Definition 2.8 is valid for R^{-s} and all weights γ varying in finite weight strips of prescribed finite widths we can form $R^{-s}(r_k^{-e}\mathcal{K}^{0;\beta,\gamma_k+e}(B^\wedge)) = r_k^{-e}\tilde{R}^{-s}\mathcal{K}^{0;\beta,\gamma_k+e}(B^\wedge)$ where $\tilde{R}^{-s}$ is order reducing with a shifted Mellin symbol, cf. analogously Theorem 1.3 (iv). Thus it suffices to apply Corollary 2.11.

Theorem 2.14. *For every $s' \geq s$, $\gamma' \geq \gamma$ (component wise) and $e' \geq e$ we have a continuous embedding*

$$\iota_s : \mathcal{K}^{s',\gamma';e'}(B^\wedge) \hookrightarrow \mathcal{K}^{s,\gamma;e}(B^\wedge), \tag{2.40}$$

and (2.40) *is compact for $s' > s$, $\gamma' > \gamma$, $e' > e$.*

Proof. We have

$$\mathcal{K}^{s',\gamma';e'}(B^\wedge) = R^{-s'}\mathcal{K}^{0,\gamma'-s';e'}(B^\wedge), \quad \mathcal{K}^{s,\gamma;e}(B^\wedge) = R^{-s}\mathcal{K}^{0,\gamma-s;e}(B^\wedge)$$

and thus

$$\mathcal{K}^{s',\gamma';e'}(B^\wedge) = R^{-s}(R^sR^{-s'})\mathcal{K}^{0,\gamma'-s';e'}(B^\wedge) = R^{-s}\mathcal{K}^{-(s-s'),\gamma'-s'-(s-s');e'}(B^\wedge)$$

Because of the continuous/compact embedding

$$\iota_0 : \mathcal{K}^{-(s-s'),\gamma'-s;e'}(B^\wedge) \hookrightarrow \mathcal{K}^{0,\gamma-s\,;e}(B^\wedge)$$

it follows that

$$\iota_s : R^{-s}\mathcal{K}^{-(s-s'),\gamma'-s;e'}(B^\wedge) \hookrightarrow R^{-s}\mathcal{K}^{0,\gamma-s\,;e}(B^\wedge)$$

is continuous/compact, as claimed. Note that in the latter conclusion we inductively employed continuous/compact embeddings $H^{s',\beta'}(B) \hookrightarrow H^{s,\beta}(B)$ for $s' \geq s$, $\gamma' \geq \gamma$ and $s' > s$, $\beta' > \beta$, respectively. $\square$

The spaces $\mathcal{K}^{s;\beta,\gamma_k}(B^\wedge)$ are Hilbert spaces with group action $\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ for (1.30), since κ acts on $\mathcal{K}^{0,\gamma-s}(B^\wedge)$, and the shape of R^{-s} shows how κ acts on $\mathcal{K}^{0,\gamma-s}(B^\wedge)$. We can define associated wedge spaces

$$\mathcal{W}^s(\mathbb{R}^{q_k}, \mathcal{K}^{s,\gamma}(B^\wedge))$$

and according to (2.9) establish analysis on the next singularity level, e.g., form spaces $H^{s,\gamma}(M)$ for any $M \in \mathfrak{M}_k$, near $s_k(M)$ modeled on $B^\Delta \times \mathbb{R}^{q_k}$ for $B \in \mathfrak{M}_{k-1}$.

Theorem 2.15. *An operator $A \in L^\mu(M, \boldsymbol{g})$ for $M \in \mathfrak{M}_k$ induces continuous operators*

$$A : H^{s,\gamma}(M) \to H^{s-\mu,\gamma-\mu}(M) \tag{2.41}$$

for all $s \in \mathbb{R}$ and $\gamma = (\gamma_1, \ldots, \gamma_k)$ involved in $\boldsymbol{g}$. If $\sigma_j(A) = 0$ for $j = 0, \ldots, k$ then (2.41) *is compact.*

Proof. The spaces $H^{s,\gamma}(M)$ are defined in an analogous manner as (2.9) and $L^\mu(M,\boldsymbol{g})$ locally near $s_k(M)$ consists of all $\mathrm{Op}_{y_k}(a)$ for $a(y_k,\eta_k) \in R^\mu(\mathbb{R}^{q_k} \times \mathbb{R}^{q_k},\boldsymbol{g})$, cf. Definition 2.1. Because of

$$R^\mu(\mathbb{R}^{q_k} \times \mathbb{R}^{q_k},\boldsymbol{g}) \subset S^\mu(\mathbb{R}^{q_k} \times \mathbb{R}^{q_k};\mathcal{K}^{s,\gamma}(B^\wedge),\mathcal{K}^{s-\mu,\gamma-\mu}(B^\wedge)).$$

This entails the continuity of

$$\mathrm{Op}_{y_k}(a) : \mathcal{W}^s(\mathbb{R}^{q_k},\mathcal{K}^{s,\gamma}(B^\wedge)) \to \mathcal{W}^{s-\mu}(\mathbb{R}^{q_k},\mathcal{K}^{s-\mu,\gamma-\mu}(B^\wedge))$$

which is the main contribution to the continuity of (2.41). The contribution from the "int"-part of A is known by the iterative step before. □

2.4. The Calculus for Singular Cones

Operators on singular cones are an aspect of the edge symbolic calculus, analogously as operators on the half-axis normal to a boundary appearing as boundary symbols in boundary value problems. On singular spaces $M \in \mathfrak{M}_k$ in general for convenience we assumed that M is compact. However, the symbolic structure of operators living on M requires considering infinite cones B^Δ for $B \in \mathfrak{M}_{k-1}$ which have difficult conical exits to infinity. Therefore, we establish here as a tool the parameter-dependend operator classes

$$L^\mu(B^\wedge,\boldsymbol{g};\mathbb{R}^d \setminus \{0\}) \tag{2.42}$$

living on an open stretched cone $B^\wedge$ for a $B \in \mathfrak{M}_{k-1}$. It would be more consequent to write $L^\mu(B^\Delta,\boldsymbol{g};\mathbb{R}^d)$ rather than (2.42) but our notation points out the specific aspect of interpreting $r_k \to \infty$ as a conical exit to ∞ which may contribute additional exit symbols. However, we first concentrate on asymptotic effects and Green and Mellin operators with meromorphic Mellin symbols. Later on in this subsection we comment the nature of operators on the infinite cone from the point of view of other interesting aspects.

Weight data $\boldsymbol{g} := (\gamma_i,\gamma_i-\mu,\Theta_i)_{i=1,\dots,k}$ for weight intervals $\Theta_i = (-(\vartheta_i+1),0]$, $\vartheta_i \in \mathbb{N}$, in the case $B =: X \in \mathfrak{M}_0$ only consist of one component; before we often wrote β rather than γ_1. Let us start the consideration with Green operators and Mellin operators with asymptotics. First we establish discrete asymptotic types and singular functions. A discrete asymptotic type associated with weight data (α,Θ) and dimension $b := \dim B$, for a weight $\alpha \in \mathbb{R}$ and a weight interval $\Theta = (-(\vartheta+1),0]$ for a $\vartheta \in \mathbb{N} \cup \{\infty\}$ is a sequence

$$\mathcal{P} := \{(p_j,m_j)\}_{j=1,\dots,N} \subset \mathbb{C} \times \mathbb{N} \tag{2.43}$$

such that $\pi_{\mathbb{C}}\mathcal{P} := \{p_j\}_{j=1,\dots,N}$ is finite if ϑ is finite, otherwise $\mathrm{Re}\, p_j \to \infty$ for $j \to \infty$ and

$$\pi_{\mathbb{C}}\mathcal{P} \subset \{v \in \mathbb{C} : (b+1)/2 - \alpha + \vartheta < \mathrm{Re}\, v < (b+1)/2 - \alpha)\}.$$

Definition 2.16. The space of singular functions $\mathcal{E}_{\mathcal{Q},\mathcal{P}_k}(B^\wedge)$ over $B^\wedge$, $B \in \mathfrak{M}_{k-1}$, for a finite asymptotic type $\mathcal{P}_k$ associated with the weight data (γ_k, Θ_k) is defined as the set

$$\mathcal{E}_{\mathcal{Q},\mathcal{P}_k}(B^\wedge) := \{\omega_k \sum_{j=0}^{N} \sum_{l=0}^{m_j} c_{jl} r_k^{-p_j} \log^l r_k : c_{jl} \in H_{\mathcal{Q}}^{\infty,\beta}(B)\}, \tag{2.44}$$

where $\omega_k = \omega_k(r_k)$ is a cut-off function and $\mathcal{Q} := (\mathcal{Q}_1, \ldots, \mathcal{Q}_{k-1})$ is a tuple of asymptotic types with $\mathcal{Q}_j$ being associated with the weight data (γ_j, Θ_j), $j = 1, \ldots, k-1$.

Definition 2.16 is inductive, and for $k = 1$ and $X^\wedge$ instead of $B^\wedge$ for $X \in \mathfrak{M}_0$ we recover the case treated in [53] or [56]. The spaces $\mathcal{E}_{\mathcal{Q},\mathcal{P}_k}(B^\wedge)$ are Fréchet in a natural way, and similarly as for $k = 1$ we can form the spaces

$$\mathcal{K}_{\mathcal{Q},\Theta_k}^{s;\beta,\gamma_k}(B^\wedge) := \big(\varprojlim_{0<\varepsilon<\vartheta_k+1} \mathcal{K}^{s;\beta,\gamma_k-\vartheta_k-\varepsilon}(B^\wedge)\big) \cap \mathcal{K}_{\mathcal{Q}}^{s,\beta}((2\mathbb{B})^\wedge)|_{(B\setminus s_{k-1}(B))^\wedge}, \tag{2.45}$$

as well as the non-direct sum

$$\mathcal{K}_{\mathcal{Q},\mathcal{P}_k}^{s;\beta,\gamma_k}(B^\wedge) := \mathcal{K}_{\mathcal{Q},\Theta_k,}^{s;\beta,\gamma_k}(B^\wedge) + \mathcal{E}_{\mathcal{Q},\mathcal{P}_k}(B^\wedge). \tag{2.46}$$

In future we apply again notation $\gamma = (\beta, \gamma_k)$ and we write $\mathcal{P}$ rather than $(\mathcal{Q}, \mathcal{P}_k)$, and we also form the spaces $\mathcal{K}_{\mathcal{P}}^{s,\gamma;e}(B^\wedge) := [r_k]^{-e}\mathcal{K}_{\mathcal{P}}^{s,\gamma}(B^\wedge)$ for any $e \in \mathbb{R}$. The spaces $H_{\mathcal{Q}}^{\infty,\beta}(B)$ occurring in (2.44) are also defined in an iterative manner, using $\mathcal{W}^s(\mathbb{R}^{q_{k-1}}, \mathcal{K}_{\mathcal{Q}}^{s,\beta}(C^\wedge))$, where B is locally near $s_{k-1}(B)$ modeled on $C^\Delta \times \mathbb{R}^{q_{k-1}}$ for some $C \in \mathfrak{M}_{k-2}$.

Definition 2.17. An element $g(y_k, \eta_k) \in S_{\mathrm{cl}}^{\mu}(\mathbb{R}^{q_k} \times \mathbb{R}^{q_k}; \mathcal{K}^{s,\gamma}(B^\wedge), \mathcal{K}^{\infty,\gamma-\mu}(B^\wedge))$ is called a Green symbol if

$$g(y_k, \eta_k) \in S_{\mathrm{cl}}^{\mu}(\mathbb{R}^{q_k} \times \mathbb{R}^{q_k}; \mathcal{K}^{s,\gamma;e}(B^\wedge), \mathcal{K}_{\mathcal{P}}^{\infty,\gamma-\mu;\infty}(B^\wedge)), \tag{2.47}$$

$$g^*(y_k, \eta_k) \in S_{\mathrm{cl}}^{\mu}(\mathbb{R}^{q_k} \times \mathbb{R}^{q_k}; \mathcal{K}^{s,-\gamma+\mu;e}(B^\wedge), \mathcal{K}_{\mathcal{S}}^{\infty,-\gamma;\infty}(B^\wedge)) \tag{2.48}$$

for all $s, e \in \mathbb{R}$, where "g^*" indicates the $\mathcal{K}^{0,0}(B^\wedge)$-adjoint of g, and $\mathcal{P} = (\mathcal{P})_{k=1,\ldots,k}$, $\mathcal{S} = (\mathcal{S})_{k=1,\ldots,k}$, are asymptotic types depending on g.

Remark 2.18. *By definition Green symbols exist on every singular level, in particular, for $k-1$, and we can formally replace the covariable η_k by (η_k, ζ). As such they generate the operator class $L_{\mathrm{G}}^{\mu}(B, \boldsymbol{g}; \mathbb{R}^d) \subset L^{\mu}(B, \boldsymbol{g}; \mathbb{R}^d)$ with notation as in Subsection* 1.5. *If we drop (y_k, η_k) at all, then instead of Green symbols as in Definition* 2.17 *we simply obtain parameter-dependent operators*

$$G(\zeta) : \mathcal{K}^{s,\gamma;e}(B^\wedge) \to \mathcal{K}_{\mathcal{P}}^{\infty,\gamma-\mu;\infty}(B^\wedge), \tag{2.49}$$

$$G^*(\zeta) : \mathcal{K}^{s,\gamma;e}(B^\wedge) \to \mathcal{K}_{\mathcal{S}}^{\infty,\gamma-\mu;\infty}(B^\wedge). \tag{2.50}$$

Those furnish the subclass

$$L_{\mathrm{G}}^{\mu}(B^\wedge, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\}) \subset L^{\mu}(B^\wedge, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\})$$

of Green operators on the singular cone $B^\wedge$.

Let us now define other subclasses, called smoothing Mellin plus Green operators. Those are also important both over B as well as over $B^{\wedge}$. Let us first give a definition of asymptotic types for smoothing Mellin symbols. Such an asymptotic type is a sequence

$$\mathcal{R} := \{(r_j, n_j)\}_{j\in\mathbb{Z}} \subset \mathbb{C}\times\mathbb{N}$$

such that $\pi_{\mathbb{C}}\mathcal{R} := \{(r_j)\}_{j\in\mathbb{Z}}$ intersects every strip $\{v \in \mathbb{C} : c \le \operatorname{Re} v \le c'\}$ for finite $c \le c'$ in a finite set of points.

Definition 2.19. By $M_{\mathcal{R}}^{-\infty}(B, \boldsymbol{g}_B)$ we denote the set of all

$$f \in \mathcal{A}(\mathbb{C}\setminus\pi_{\mathbb{C}}\mathcal{R}, L^{-\infty}(B, \boldsymbol{g}_B))$$

which are meromorphic with poles at the points of $\pi_{\mathbb{C}}\mathcal{R}$ of multiplicity $n_j + 1$ and Laurent coefficients of finite rank belonging to $L^{-\infty}(B, \boldsymbol{g}_B)$. In addition it is required that for every $\pi_{\mathbb{C}}\mathcal{R}$ -excision function χ we have $\chi f|_{\Gamma_\lambda} \in \mathcal{S}(\Gamma_\lambda, L^{-\infty}(B, \boldsymbol{g}_B))$ for every λ, uniformly for compact λ-intervals.

Smoothing Mellin operators associated with the weight interval $(-(\vartheta_k + 1), 0]$ for $\vartheta_k \in \mathbb{N}$ on the singularity level $k \in \mathbb{N}$ are written in the form

$$M := r_k^{-\mu}\omega_{k,\zeta}\sum_{j=0}^{\vartheta_k}\sum_{|\alpha|\le j} r_k^j \mathrm{Op}_M^{\gamma_{k,j\alpha}-b/2}(f_{j\alpha})\zeta^\alpha\omega'_{k,\zeta} \tag{2.51}$$

for $\omega_{k,\zeta}(r_k) = \omega_k(r_k[\zeta])$ with ω_k being an excision function, arbitrary $f_{j\alpha}(v_k) \in M_{\mathcal{R}_{j\alpha}}^{-\infty}(B, \boldsymbol{g}_B)$ for Mellin asymptotic types $\mathcal{R}_{j\alpha}$ and weights

$$\gamma_k - j \le \gamma_{k,j\alpha} \le \gamma_k \quad \text{such that} \quad \mathcal{R}_{j\alpha}\cap\Gamma_{(b+1)/2} - \gamma_{k,j\alpha} = \emptyset.$$

By $L_{\mathrm{M+G}}^\mu(B^{\wedge}, \boldsymbol{g}; \mathbb{R}^d\setminus\{0\}) \subset L^\mu(B^{\wedge}, \boldsymbol{g}; \mathbb{R}^d\setminus\{0\})$ we denote the space of all operators $M + G$ for arbitrary M of the form (2.51) and $G \in L_{\mathrm{G}}^\mu(B^{\wedge}, \boldsymbol{g}; \mathbb{R}^d\setminus\{0\})$. By modifying notation in (2.51) we obtain operator-valued symbols

$$m(y_k, \eta_k) := r_k^{-\mu}\omega_{k,\eta_k}\sum_{j=0}^{\vartheta_k}\sum_{|\alpha|\le j} r_k^j \mathrm{Op}_M^{\gamma_{k,j\alpha}-b/2}(f_{j\alpha})(y_k)\eta_k^\alpha\omega'_{k,\eta_k} \tag{2.52}$$

for arbitrary $f_{j\alpha}(y_k, v_k) \in C^\infty(\Omega_k, M_{\mathcal{R}_{j\alpha}}^{-\infty}(B, \boldsymbol{g}_B))$, $\Omega_k \subseteq \mathbb{R}^{q_k}$ open. Symbols $(m + g)(y_k, \eta_k)$ form a subspace $R_{\mathrm{M+G}}^\mu(\mathbb{R}^{q_k}\times\mathbb{R}^{q_k}, \boldsymbol{g}) \subset R^\mu(\Omega_k\times\mathbb{R}^{q_k}, \boldsymbol{g})$ constituting the subclass

$$L_{\mathrm{M+G}}^\mu(M, \boldsymbol{g}; \mathbb{R}^d) \subset L^\mu(M, \boldsymbol{g}; \mathbb{R}^d)$$

of smoothing Mellin plus Green operators in the calculus over $M \in \mathfrak{M}_k$. Operators of the latter class play a similar role as those in the lower singular calculus, cf. [53] or [56], and they have similar properties. For $M = \mathrm{Op}_{y_k}(m)$, $m(y_k, \eta_k)$ as in (2.52) we define

$$\sigma_k(M)(y_k, \eta_k) := r_k^{-\mu}\omega_{k,|\eta_k|}\sum_{j=0}^{\vartheta_k}\sum_{|\alpha|=j} r_k^j \mathrm{Op}_M^{\gamma_{k,j\alpha}-b/2}(f_{j\alpha})(y_k)\eta_k^\alpha\omega'_{k,|\eta_k|}$$

for $\omega_{k,|\eta_k|}(r_k) := \omega_k(r_k|\eta_k|)$, etc.

The operator space (2.42) with parameter $\zeta \in \mathbb{R}^d \setminus \{0\}$) is defined as the set of all operator families

$$A(\zeta) = H(\zeta) + (M+G)(\zeta) + C(\zeta) \tag{2.53}$$

for $(M+G)(\zeta) \in L^{\mu}_{\mathrm{M+G}}(B^\wedge, \boldsymbol{g}; \mathbb{R}^d \setminus \{0\})$ and

$$H(\zeta) := r_k^{-\mu} \mathrm{Op}_M^{\gamma_k - b/2}(h)(\zeta) \tag{2.54}$$

for arbitrary $h(r_k, v_k, \zeta) \in M^{\mu}_{\mathcal{O}_{v_k}}(B, \boldsymbol{g}_B; \mathbb{R}^d_{r_k \zeta})$.

Theorem 2.20. *Operators* $A(\zeta)$ *in* (2.42) *induce continuous operators*

$$A(\zeta) : \mathcal{K}^{s,\gamma}(B^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(B^\wedge) \tag{2.55}$$

and

$$A(\zeta) + (M+G)(\zeta) : \mathcal{K}^{s,\gamma}_{\mathcal{P}}(B^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}_{\mathcal{Q}}(B^\wedge) \tag{2.56}$$

for every $\mathcal{P}$ *with some resulting* $\mathcal{Q}$.

Proof. The continuity of $A(\zeta)$ in (2.55) is a consequence of the defintion of spaces $\mathcal{K}^{s,\gamma}(B^\wedge)$, cf. Definition 2.8. The continuity of $G(\zeta)$ in (2.55), more precisely, of

$$G(\zeta) : \mathcal{K}^{s,\gamma}(B^\wedge) \to \mathcal{K}^{\infty,\gamma-\mu}(B^\wedge)$$

is part of the definition of Green operators, and the continuity of $M(\zeta)$ in (2.55), in fact, $M(\zeta) : \mathcal{K}^{s,\gamma}(B^\wedge) \to \mathcal{K}^{\infty,\gamma-\mu}(B^\wedge)$ can be reduced to order zero by composing with reductions of orders that are involved in Definition 2.8. The order reductions preserve smoothing Mellin plus Green operators, and the result for $s = 0$ is a simple consequence. The claimed continuity of $A(\zeta)$ in (2.56) in spaces $\mathcal{K}^{s;\beta\gamma_k}_{\mathcal{Q},\Theta_k}(B^\wedge)$ is a consequence of the former continuity on the flat spaces on the right-hand side of (2.45) combined with commutation of powers of r_k through the Mellin action, see relation (2.39) and the higher analogue of Theorem 1.3 (iv). Moreover, singular functions in (2.44) are mapped via the Mellin transform to meromorphic functions with $H^{\infty,\beta}_{\mathcal{Q}}(B)$-valued Laurent coefficients. The multiplication by holomorphic Mellin symbols involved in $A(\zeta)$ gives us again such meromorphic functions. Subsequent application of the inverse Mellin transform gives us back such asymptotic terms in r_k, except for the cut-off factor. However, decomposing the identity into $\sigma_k + (1-\sigma_k)$ the summand containing σ_k is of the type of a singular function, while the summand with $(1-\sigma_k)$ just belongs to (2.45) for $s = -\infty$. Thus the continuity of $A(\zeta)$ in (2.46) is verified. For $G(\zeta)$ the claimed continuity is clear by definition, while for $M(\zeta)$ we can argue for each summand separately. The arguments are the analogous as for $A(\zeta)$ in (2.46). □

2.5. Operators of Third Singularity Order

Let us now analyze the structure of operators in $M \in \mathfrak{M}_k$ for $k = 3$,

$$M := B^{\Delta}_{k-1} \times \mathbb{R}^{q_k} \ni (r_k, x_{k-1}, y_k),\ B_{k-1} \in \mathfrak{M}_{k-1}. \tag{2.57}$$

Here, in abuse of notation, we write coordinates in stretched form, i.e., from $B^{\wedge}_{k-1}\times\mathbb{R}^{q_{k-1}}$. We successively assume

$$B_{k-1} := B^{\Delta}_{k-2}\times\mathbb{R}^{q_{k-1}} \ni (r_{k-1},x_{k-2},y_{k-1}),\ B_{k-2}\in\mathfrak{M}_{k-2},$$
$$B_{k-2} := B^{\Delta}_{k-3}\times\mathbb{R}^{q_{k-2}} \ni (r_{k-2},x_{k-3},y_{k-2}),\ B_{k-3}\in\mathfrak{M}_{k-3},$$

etc. The end of the iteration is given by the space

$$B_1 = X^{\Delta}\times\mathbb{R}^{q_1}\ni(r_1,x_0,y_1),\ X\in\mathfrak{M}_0$$

for $x_0 =: x\in X$, and X compact.

Similarly as for corner-degenerate differential operators of Subsection 2.1 where $k=2$, for any $D\in L^{\mu}(M,\boldsymbol{g}_k)$ we first have the principal symbol of k-th order

$$\sigma_k(D)(y_k,\eta_k). \tag{2.58}$$

But then, for the lower order principal symbols, we look at the configuration off $r_k=0$ such that $(r_k,y_k)\in\mathbb{R}_{+,r_k}\times\mathbb{R}^{q_k}$ become additional edge-variables. Thus,

$$\sigma_{k-1}(D)(r_k,y_{k-1},y_k,\rho_k,\eta_{k-1},\eta_k). \tag{2.59}$$

In this case ρ_k has the meaning of the Fourier covariable of the edge coordinate $r_k\in\mathbb{R}_+$. In other words, not (y_{k-1},η_{k-1}) form variables and covariables on the edge of order $k-1$ but $(r_k,y_{k-1},y_k,\rho_k,\eta_{k-1},\eta_k)$. In a similar manner, not (y_{k-j},η_{k-j}) for $j>1$ are variables and covariables of the edge of order $k-j$, but variables and covariables in

$$\begin{aligned}\sigma_{k-j}(D)((r_{k-j+1},y_{k-j+1}),\dots,(r_{k-1},y_{k-1}),r_k,y_k,&\\ (\rho_{k-j+1},\eta_{k-j+1}),\dots,(\rho_{k-1},\eta_{k-1}),\rho_k,\eta_k).&\end{aligned} \tag{2.60}$$

At the end of the iteration it follows that

$$\begin{aligned}\sigma_0(D)((r_1,x,y_1),\dots,(r_{k-j},y_{k-j}),\dots,(r_{k-1},y_{k-1}),y_k,(\rho_1,\xi,\eta_1),\dots,&\\ (\rho_{k-j},\eta_{k-j}),\dots,(\rho_{k-1},\eta_{k-1}),\eta_k).&\end{aligned} \tag{2.61}$$

$$A(\zeta)=r_k^{-\mu}\mathrm{Op}_M^{\gamma_k-b/2}(h)(\zeta)\quad\text{for}\quad h(r_k,v_k,\zeta)\in M^{\mu}_{\mathcal{O}_{v_k}}(B,\boldsymbol{g}_\beta;\mathbb{R}^d_{r_k\zeta}).$$

We now consider a pseudo-differential example of operators in $L^{\mu}(B^{\wedge},\boldsymbol{g},\mathbb{R}^d_{\zeta})$ for, say, $B\in\mathfrak{M}_2$; then $B^{\Delta}\in\mathfrak{M}_3$. We consider the case $d:=q_3$, $\zeta:=\eta_3$ and

$$B^{\Delta}\times\mathbb{R}^{q_3}=\Big(\big(B_1^{\Delta}\times\mathbb{R}^{q_1}\big)^{\Delta}\times\mathbb{R}^{q_2}\Big)^{\Delta}\times\mathbb{R}^{q_3}.$$

The operators $D_3\in L^{\mu}(B^{\Delta}\times\mathbb{R}^{q_3},\boldsymbol{g})$ have third order edge symbols of the form

$$A_3 := r_3^{-\mu}\mathrm{Op}_{M_{r_3}}^{\gamma_3-b_2/2}(h_0)(y_3,\eta_3) \tag{2.62}$$

with $r_3^{-\mu}$ being regarded as part of the operation

$$A_3:\mathcal{K}^{s,\gamma_4}(B^{\wedge})\to\mathcal{K}^{s-\mu,\gamma_4-\mu}(B^{\wedge}).$$

For convenience we consider operators

$$D_3 := \varphi_3\mathrm{Op}_{y_3}\big\{\sigma_3 r_3^{-\mu}\mathrm{Op}_{M_{r_3}}^{\gamma_3-b_2/2}(f_3)(y_3,\eta_3)\sigma_3'\big\}\varphi_3', \tag{2.63}$$

for $b_2 := \dim B_1 + 1 + q_1$ and $B^\wedge$ with (third order) edge $\mathbb{R}^{q_3} \ni y_3$, where $\varphi_3,\ \varphi_3' \in C_0^\infty(\mathbb{R}^{q_3})$, $\varphi_3 \prec \varphi_3'$, and $\sigma_3,\ \sigma_3'$ are cut-off functions on the $\mathbb{R}_{+,r_3}$ half-axis, $\sigma_3 \prec \sigma_3'$. Moreover, $f(r_3, y_3, v_3, \eta_3) := \tilde{f}(r_3, y_3, v_3, \tilde{\eta}_3)|_{\tilde{\eta}_3 = r_3\eta_3}$, $\tilde{f}(r_3, y_3, v_3, \tilde{\eta}_3) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^{q_3}, M_{\mathcal{O}}^\mu(B, \boldsymbol{g}_\beta; \mathbb{R}^{q_k}_{\tilde{\eta}_3}))$, and

$$h_0(r_3, y_3, v_3, \eta_3) = \tilde{f}(0, y_3, v_3, r_3\eta_3).$$

The absence of cut-off functions in (2.62) is explained by the structure of edge symbols where ζ plays the role of the edge covariable $\eta_3 \neq 0$ and the first r_3 in $\tilde{f}$ is frozen at zero. Similarly as in Section 1 we have twisted homogeneity

$$A(\delta\eta_3) = \delta^\mu \kappa_\delta A(\eta_3) \kappa_\delta^{-1} \quad \text{for} \quad \delta \in \mathbb{R}_+. \tag{2.64}$$

Observe that for cut-off functions $\omega_3 \prec \omega_3'$ in r_3 the operator families

$$\omega_3 A(\eta_3)(1 - \omega_3') \quad \text{and} \quad (1 - \omega_3') A(\eta_3) \omega_3$$

are homogeneous Green symbols. Recall that in the definition of Kegel spaces we imposed a specific ellipticity which is not typical for our Mellin symbols in general. But we want to illustrate the iterative structure and the higher corner-degenerate behaviour. The position of lower order degenerate Mellin symbols within h is as follows. We can apply the principle of sleeping parameters which are successively wakened to induce higher singular operators. Let us assume in our example that

$$f_3(r_3, y_3, v_3, \eta_3)$$

consists of parameter-dependent operators of the form

$$\varphi_2 \mathrm{Op}_{y_2}\big\{\sigma_2 r_2^{-\mu} \mathrm{Op}_{M_{r_2}}^{\gamma_2 - b_2/2}(f_2)(y_2, \eta_2)\sigma_2'\big\}\varphi_2', \tag{2.65}$$

for

$$f_2(r_2, y_2, v_2, \eta_2) := \tilde{f}_2(r_2, y_2, v_2, \tilde{\eta}_2)|_{\tilde{\eta}_2 = r_2\eta_2} \tag{2.66}$$

$$\tilde{f}_2(r_2, y_2, v_2, \tilde{\eta}_2) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^{q_2}, M_{\mathcal{O}_{v_2}}^\mu(B_1, \boldsymbol{g}_1; \mathbb{R}^{q_2}_{\tilde{\eta}_2})) \tag{2.67}$$

where $\varphi_2,\ \varphi_2' \in C_0^\infty(\mathbb{R}^{q_2})$, $\varphi_2 \prec \varphi_2'$, and $\sigma_2,\ \sigma_2'$ are cut off functions on the $\mathbb{R}_{+,r_2}$ half-axis, $\sigma_2 \prec \sigma_2'$. Moreover, we assume that

$$f_2(r_2, y_2, v_2, \eta_2)$$

consists of parameter-dependent operators of the form

$$\varphi_0 \mathrm{Op}_x \varphi_1 \mathrm{Op}_{y_1}\big\{\sigma_1 r_1^{-\mu} \mathrm{Op}_{M_{r_1}}^{\gamma_1 - n/2}(f_1)(y_1, \eta_1)\sigma_1'\big\}\varphi_1'\varphi_0' \tag{2.68}$$

for

$$f_1(r_1, x, \xi, y_1, v_1, \eta_1) := \tilde{f}_1(r_1, x, \xi, y_1, v_1, \tilde{\eta}_1)|_{\tilde{\eta}_1 = r_1\eta_1} \tag{2.69}$$

$$\tilde{f}_1(r_1, y_1, v_1, \tilde{\eta}_1) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^{q_1}, M_{\mathcal{O}_{v_1}}^\mu(X; \mathbb{R}^{q_1}_{\tilde{\eta}_1})) \tag{2.70}$$

where $\varphi_1,\,\varphi_1' \in C_0^\infty(\mathbb{R}^{q_1}), \varphi_0,\,\varphi_0' \in C_0^\infty(\mathbb{R}^n)$ for $n = \dim X$, $\varphi_1 \prec \varphi_1'$, $\varphi_0 \prec \varphi_0'$, and $\sigma_1,\,\sigma_1'$ are cut-off functions on the $\mathbb{R}_{+,r_1}$ half-axis, $\sigma_1 \prec \sigma_1'$. Then it follows altogether

$$
\begin{aligned}
D_3 = \varphi_0 \mathrm{Op}_x \varphi_1 \mathrm{Op}_{y_1} \{\sigma_1 r_1^{-\mu} \mathrm{Op}_{M_{r_1}}^{\gamma_1 - n/2} \varphi_2 \mathrm{Op}_{y_2} \{\sigma_2 r_2^{-\mu} \mathrm{Op}_{M_{r_2}}^{\gamma_2 - b_2/2} \\
\varphi_3 \mathrm{Op}_{y_3} \{\sigma_3 r_3^{-\mu} \mathrm{Op}_{M_{r_3}}^{\gamma_3 - b_2/2}(m_3) \sigma_3' \varphi_3'\} \sigma_2'\} \varphi_2' \sigma_1'\} \varphi_1' \varphi_0'
\end{aligned}
\tag{2.71}
$$

for the corner-degenerate Mellin-Fourier-symbol

$$m := m(r_1, r_2, r_3, x, y_1, y_2, y_3, \xi, v_1, r_1 v_2, r_1 r_2 v_3, r_1 \eta_1, r_1 r_2 \eta_2, r_1 r_2 r_3 \eta_3),$$

where v_i in the simplest case stands for $i\varrho_i$ in the complex v_i-plane, corresponding to weights $\gamma_i = 0$ for all i.

More general examples of elements in $M_{\mathcal{O}}^\mu(B, \boldsymbol{g}_\beta, \mathbb{R}^{q_k}_{r_k \eta_k})$, also for higher k are obtained together with kernel cut-offs as sums of expressions of the above-mentioned kind, when the functions φ_l vary over a partition of unity on $s_l(B)$ and $\varphi_l' \succ \varphi_l$ are smooth functions of compact support. By admitting extra parameters ι_k of the same degenerate behaviour as η_k, and also for the involved lower singular levels, the emerging functions $h(\iota_k, r_k, v_k, \eta_k)$ induce families of continuous operators

$$h(\iota_k, r_k, v_k, \eta_k) : H^{s,\beta}(B) \to H^{s-\mu,\beta-\mu}(B). \tag{2.72}$$

Then

$$\|(D_{r_k}^m h)(\iota_k, r_k, v_k, \eta_k)\|_{\mathcal{L}(H^{s,\beta}(B), H^{s-\mu,\beta-\mu}(B))} \le c_\alpha \langle r_k \rangle^{-m} \tag{2.73}$$

for every $m \in \mathbb{N}$. If we briefly set $H := H^{s,\beta}(B)$, $\widetilde{H} := H^{s-\mu,\beta-\mu}(B)$ we obtain

$$S^{\mu;0}(\mathbb{R}_+ \times \mathbb{R}; H, \widetilde{H}) \tag{2.74}$$

which is a space of operator-valued symbols with interior order μ and exit order 0 for $r \to \infty$. Here $\mathbb{R}$ is the space of covariables and can be interpreted as a parallel to the imaginary axis Γ_λ in the complex plane. On the level of principal symbols with respect to $\rho \in \mathbb{R}$ we may interprete Γ_λ as Γ_{λ_0} for any real λ_0, as a consequence of the holomorphy of Mellin symbols in v_k and Cauchy's theorem. Observe that if $h(r, \rho)$ belongs to (2.74) then

$$D_r^m D_\rho^n h(r, \rho) \in S^{\mu-m;-n}(\mathbb{R}_+ \times \mathbb{R}; H, \widetilde{H}).$$

Also the mapping properties of operators in $\mathcal{L}(H, \widetilde{H})$ becomes better under differentiation, e.g., the resulting operators become compact for $m > 0, n > 0$.

$$\mathcal{A} = \begin{pmatrix} A & K \\ T & Q \end{pmatrix} : \begin{matrix} H^{s,\gamma}(M, E) \\ \oplus \\ H^s(Y, J) \end{matrix} \to \begin{matrix} H^{s-\mu,\gamma-\mu}(M, F) \\ \oplus \\ H^{s-\mu}(Y, G) \end{matrix}. \tag{2.75}$$

The motivation of such operator block matrices comes from elliptic theory. Assuming that A is σ_0-elliptic, i.e., that

$$\sigma_0(A) : \pi^* E \to \pi^* F$$

for $\pi : T^*(s_0(M)) \setminus 0 \to s_0(M)$ is an isomorphism and also the reduced symbol close to Y in the local splitting of variables (r, x, y) with covariables (ρ, ξ, η) defined by $\tilde{\sigma}_0(A)(r, x, y, \rho, \xi, \eta) := r^{\mu}\sigma_0(A)(r, x, y, r^{-1}\rho, \xi, r^{-1}\eta)$

$$\tilde{\sigma}_0(A)(r, x, y, \rho, \xi, \eta) : E_{(r,x,y)} \to F_{(r,x,y)}$$

for all $(\rho, \xi, \eta) \neq 0$, up to $r = 0$.

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