On Multi-Pointed Non-Commutative Deformations and Calabi-Yau Threefolds

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1. INTRODUCTION

We shall develop a theory of multi-pointed non-commutative deformations of a simple collection in an abelian category. A simple collection is a finite set of objects such that each object has no endomorphisms except dilations and there are no nonzero homomorphisms between objects. We note that the objects in a simple collection are not necessarily simple objects. The commutative deformations of several objects are just the direct product of deformations of each objects, but there are interactions of objects in the case of non-commutative deformations. We shall prove that any iterated non-trivial extensions between the given objects yields a non-commutative deformation in the case of a simple collection, and we obtain a versal deformation in this way. As applications, we will construct relative exceptional objects and relative spherical objects in some special cases.

The deformation theory has non-commutative versions in two directions, non-commutative fibers and non-commutative base. We consider the latter case. The point is that there are more non-commutative deformations of commutative objects than the commutative deformations as proved in a paper by Donovan and Wemyss [4]. They discovered an interesting application of the theory of non-commutative deformations to the theory of three dimensional algebraic varieties. They provided a better understanding of the mysterious analytic neighborhood of a flopping curve on a threefold by investigating non-commutative deformations of the flopping curve. The invariants defined by them are found to be related to Gopakumar-Vafa invariants and Donaldson-Thomas invariants ([11]). This paper is motivated by their works. Moreover we consider systematically multi-pointed deformations, i.e., non-commutative deformations of several objects.

The theory of deformations over a non-commutative base is developed by Laudal [7]. The definition of non-commutative deformations is very similar to the commutative deformations, but only the parameter algebra is not necessarily commutative. A non-commutative Artin semi-local algebra with nilpotent Jacobson radical is not necessarily a direct product of Artin local algebras. By this reason, we need to consider several maximal ideals simultaneously.

The infinitesimal extensions of a deformation and the obstruction theory is similarly described by cohomology groups as in [8], and there exists a versal family of non-commutative deformations under some mild conditions. But there are much more non-commutative deformations than the commutative ones. For example, unobstructed deformations in the commutative case can be obstructed in the non-commutative sense.

Let k be a field, $\mathcal{A}$ a k-linear abelian category, r a positive integer, and F_i $(1 \le i \le r)$ objects in $\mathcal{A}$. The set $\{F_i\}$ is said to be a *simple collection* if $\dim \mathrm{Hom}(F_i, F_j) = \delta_{ij}$. We define non-commutative deformations of the collection $\{F_i\}$ as iterated non-trivial mutual extensions of the F_i. We shall prove that the non-commutative deformations behave very nicely under the condition of simplicity.

In §2, we define a multi-pointed non-commutative deformation of a collection of objects. In §3, we treat non-commutative deformations of objects as their iterated extensions. First theorem states that, for any two sequences of iterated non-trivial extensions of a simple collection, there exists a third sequence of iterated non-trivial extensions which dominates others (Theorem 3.3). In particular, if the extensions terminate, then there exists a unique versal deformation.

In the second theorem in §4, we prove the converse that arbitrary sequence of iterated non-trivial extensions of a simple collection can be regarded as a non-commutative deformation. The points is that the base ring of the deformation is recovered as the ring of endomorphisms. For this purpose, we consider a tower of universal extensions of a simple collection, and we prove that the flatness of the extension over the ring of endomorphisms. In this way we construct a versal multi-pointed non-commutative deformation (Theorem 4.8).

As applications we construct relative multi-pointed exceptional objects and relative multi-pointed spherical objects in some special cases in §5 and §6. A relative multi-pointed exceptional object yields a semi-orthogonal decomposition of a triangulated category, and a relative multi-pointed spherical object a twist functor. In the case of a local Calabi-Yau threefold, we shall prove that a versal non-commutative deformation of a simple collection becomes a relative spherical object if the deformations stop after a finite number of steps.

We shall use the abbreviation "NC" for non-commutative, or more precisely, not necessarily commutative in the rest of the paper.

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2. DEFINITION OF r-POINTED NC DEFORMATIONS

We give a definition of multi-pointed NC deformations. It is modified from [7] in order to adapt to our situation of deformations of sheaves. It seems that our treatment is also different from [5], because our definition works well only in the case of simple collections. See also [2].

We would like to consider infinitesimal deformations of r coherent sheaves on a variety at the same time for a positive integer r. If we consider only commutative deformations of these sheaves, then they deform independently. But NC deformations reflect interactions among the sheaves.

First we define the category of base rings of deformations according to [7]:

Definition 2.1. Let k be a base field, let r be a positive integer, and let k^r be the direct product ring. An *r-pointed k-algebra* R is an associative ring endowed with k-algebra homomorphisms

$$k^r \to R \to k^r$$

whose composition is the identity homomorphism.

Let e_i be the idempotents of R corresponding to the vectors $(0, \dots, 0, 1, 0, \dots, 0) \in k^r$ for $1 \le i \le r$, where 1 is at the i-th place. We have $\sum_{i=1}^r e_i = 1$, $e_i e_i = e_i$ and $e_i e_j = 0$ for $i \ne j$. Let $R_{ij} = e_i R e_j \subset R$. Then $R = \bigoplus_{i,j=1}^r R_{ij}$, and R can be considered as a matrix algebra (R_{ij}) such that the R_{ij} are k-vector spaces and the multiplication in R is given by k-linear homomorphisms $R_{ij} \otimes_k R_{jk} \to R_{ik}$.

Let M_i be the kernels of the surjective algebra homomorphisms $R \to k^r \to k$ for $1 \le i \le r$, where the second homomorphisms are i-th projections. These are maximal ideals and the R/M_i are simple two-sided R-modules. Let $M = \bigcap M_i$.

Definition 2.2. We define (Art_r) to be the category of r-pointed k-algebras R such that $\dim_k R < \infty$ and M is nilpotent.

If $R \in (\mathrm{Art}_r)$, then any simple right R-module is isomorphic to some R/M_i.

Definition 2.3. Let $\mathcal{A}$ be a k-linear abelian category. A set of objects $\{F_i\}_{i=1}^r$ in $\mathcal{A}$ is said to be a *collection* in this paper. Let $F = \bigoplus F_i$. An *r-pointed NC deformation* of the collection $\{F_i\}$ over $R \in (\mathrm{Art}_r)$ is a pair (F_R, ϕ) consisting of an object F_R of $\mathcal{A}$ which has a flat left R-module structure and an isomorphism $\phi : R/M \otimes_R F_R \cong F$ inducing isomorphisms $R/M_i \otimes_R F_R \cong F_i$ for all i. The *r-pointed NC deformation functor* $\mathrm{Def}_{\{F_i\}} : (\mathrm{Art}_r) \to (\mathit{Set})$ of $\{F_i\}$ is defined to be a covariant functor which sends R to the set of isomorphism classes of r-pointed NC deformations of $\{F_i\}$ over R.

For example, $\mathcal{A} = (\mathrm{coh}(X))$, the category of coherent sheaves on an algebraic variety X defined over k.

Remark 2.4. (1) There is a hull $\hat{R}$ for the functor $\mathrm{Def}_{\{F_i\}}$ under suitable conditions ([7]). If $r = 1$, then the maximal commutative quotient $(\hat{R})_{\mathrm{ab}}$ coincides with the hull of the usual commutative deformation functor. $\hat{R}$ is determined by $\mathrm{Ext}^1(F, F)$ and the Massey products $(\mathrm{Ext}^1(F, F))^{\otimes m} \to \mathrm{Ext}^2(F, F)$ for $m \ge 2$ ([7]). We shall not use these facts.

(2) NC deformations exist only over local base by definition. But Kapranov and Toda constructed globalization of NC deformations in the commutative direction ([12]).

3. ITERATED NON-TRIVIAL EXTENSIONS

We shall define the notion of a simple collection and consider its iterated non-trivial extensions. A simple collection behave well under iterated extensions.

Definition 3.1. Let $\mathcal{A}$ be a k-linear abelian category, and let r be a positive integer. A collection $\{F_i\}_{i=1}^r$ in $\mathcal{A}$ is said to be a *simple collection* if $\dim \mathrm{Hom}(F_i, F_j) = \delta_{ij}$.

If $\mathcal{A}$ is a category of coherent sheaves on a variety, then a member of a simple collection is usually called a *simple sheaf*. This is the origin of the term "simple". But we note that a simple sheaf is not necessary a simple object in the abelian category of sheaves.

We consider iterated non-trivial extensions of a simple collection $\{F_i\}_{i=1}^r$.

Definition 3.2. A sequence of *iterated non-trivial extensions* of the simple collection $\{F_i\}_{i=1}^r$ is a sequence of objects $\{G^n\}_{0\le n\le N}$ for a positive integer N with decompositions $G^n = \bigoplus_{i=1}^r G_i^n$ such that $G_i^0 = F_i$, and for each $0 \le n < N$, there are $i = i(n)$ and $j = j(n)$ such that

$$0 \to F_j \to G_i^{n+1} \to G_i^n \to 0$$

is an extension corresponding to a non-zero element of $\mathrm{Ext}^1(G_i^n, F_j)$, and $G_{i'}^{n+1} = G_{i'}^n$ for $i' \ne i$.

We prove that any two iterated non-trivial extensions are dominated by a third:

Theorem 3.3. *Let $\{F_i\}$ be a simple collection, and let G be an object. Let $0 \to F_{i_j} \to G_j \to G \to 0$ for $j = 0, 1$ be two non-trivial extensions which are not isomorphic. Then there exists a common object H with non-trivial extensions $0 \to F_{i_{1-j}} \to H \to G_j \to 0$.*

Proof. Let $\xi_j \in \mathrm{Ext}^1(G, F_{i_j})$ be non-zero elements corresponding to the given extensions. We consider exact sequences

$$\mathrm{Hom}(F_{i_{1-j}}, F_{i_j}) \to \mathrm{Ext}^1(G, F_{i_j}) \to \mathrm{Ext}^1(G_{1-j}, F_{i_j})$$

derived from ξ_{1-j}. Let $\xi'_j \in \mathrm{Ext}^1(G_{1-j}, F_{i_j})$ be the images of ξ_j by the second homomorphism. We claim that $\xi'_j \ne 0$. Indeed if $i_0 \ne i_1$, then the first term vanishes, hence $\xi'_j \ne 0$. If $i_0 = i_1$, then the image of the first homomorphism is generated by ξ_{1-j}, hence the image of ξ_j by the second homomorphism is non-zero because the two extensions are not isomorphic.

We have a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F_{i_1} & \xrightarrow{=} & F_{i_1} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_{i_0} & \longrightarrow & H & \longrightarrow & G_1 & \longrightarrow & 0 \\
 & & =\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_{i_0} & \longrightarrow & G_0 & \longrightarrow & G & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where the two horizontal short exact sequences correspond to ξ_0' and ξ_0. They are commutative by the construction of ξ_0'. By 9-lemma, we obtain the two vertical short exact sequences, which correspond to ξ_1' and ξ_1. Therefore we have constructed a common non-trivial extension H. $\square$

The maximal iterated non-trivial extension is unique if it exists:

Corollary 3.4. *Let $\{G^n\}_{0\le n\le N}$ and $\{H^m\}_{0\le m\le M}$ be two sequences of iterated non-trivial extensions of a simple collection $\{F_i\}$. Assume that $Ext^1(G^N,F_i)=Ext^1(H^M,F_i)=0$ for all i. Then $G^N\cong H^M$.*

Remark 3.5. (1) The above theorem is the reason why our theory works well only for simple collections.

(2) The sheaf G^N in the above corollary is the *versal r-pointed NC deformation* of the simple collection $\{F_i\}$ as proved in the next section.

We shall need the following in the next section:

Lemma 3.6. *Let $\{G^n\}$ with $G^n=\bigoplus_i G_i^n$ be a sequence of iterated non-trivial extensions of a simple collection $\{F_i\}$. Then $\dim Hom(G_i^n,F_j)=\delta_{ij}$ for all i,j,n.*

Proof. We proceed by induction on n. If $n=0$, then the assertion is true by the assumption of the simplicity. Suppose that we have an exact sequence

$$0\to F_j\to G_i^{n+1}\to G_i^n\to 0.$$

Then we have a long exact sequence

$$0\to \mathrm{Hom}(G_i^n,F_k)\to \mathrm{Hom}(G_i^{n+1},F_k)\to \mathrm{Hom}(F_j,F_k)\to \mathrm{Ext}^1(G_i^n,F_k)$$

for any k. If $k\ne j$, then the third term vanishes, hence the first arrow is bijective. If $k=j$, then the last arrow is injective because the extension is non-trivial. Therefore the first arrow is bijective again. Hence we conclude the proof. $\square$

4. ITERATED UNIVERSAL EXTENSIONS

We shall construct a sequence of universal extensions of a simple collection $\{F_i\}$ under the assumption that $\dim \mathrm{Ext}^1(F,F)<\infty$, and prove the existence of a versal r-pointed NC deformation.

Proposition 4.1. *Let $\{F_i\}_{r=1}^r$ be a simple collection, let $F=\bigoplus_{i=1}^r F_i$ be the sum of the collection, and set $F=F^{(0)}$ and $F_i=F_i^{(0)}$. Assume that $\dim Ext^1(F,F)<\infty$. Then there exists a sequence of universal extensions $F^{(n)}=\bigoplus_{i=1}^r F_i^{(n)}$ given by*

$$0\to \bigoplus_j Ext^1(F_i^{(n)},F_j)^*\otimes F_j\to F_i^{(n+1)}\to F_i^{(n)}\to 0$$

for each i, which is also obtained by sequence of iterated non-trivial extensions of the collection $\{F_i\}$.

Proof. We note that we have $\dim \operatorname{Ext}^1(F_i^{(n)}, F_j) < \infty$ for all i, j, n under the assumption. In general, given an object $G \in \mathcal{A}$ and j such that $\dim \operatorname{Ext}^1(G, F_j) < \infty$, a natural morphism $G \to \operatorname{Ext}^1(G, F_j)^* \otimes F_j[1]$ in the derived category $D(\mathcal{A})$ yields a universal extension

$$0 \to \operatorname{Ext}^1(G, F_j)^* \otimes F_j \to G_j \to G \to 0.$$

For $j' \neq j$, we have a commutative diagram

$$\begin{CD}
G_j @>>> \operatorname{Ext}^1(G_j, F_{j'})^* \otimes F_{j'}[1] \\
@VVV @VVV \\
G @>>> \operatorname{Ext}^1(G, F_{j'})^* \otimes F_{j'}[1].
\end{CD}$$

Hence a commutative diagram of extensions

$$\begin{CD}
0 @>>> \operatorname{Ext}^1(G_j, F_{j'})^* \otimes F_{j'} @>>> G'_{j,j'} @>>> G_j @>>> 0 \\
@. @VVV @VVV @VV{=}V \\
0 @>>> \operatorname{Ext}^1(G, F_{j'})^* \otimes F_{j'} @>>> G_{j,j'} @>>> G_j @>>> 0.
\end{CD}$$

Since $\operatorname{Hom}(F_j, F_{j'}) = 0$, we have an exact sequence

$$0 \to \operatorname{Ext}^1(G, F_{j'}) \to \operatorname{Ext}^1(G_j, F_{j'}) \to \operatorname{Ext}^1(G, F_j) \otimes \operatorname{Ext}^1(F_j, F_{j'}).$$

The injectivity of the first homomorphism implies that the second line of the above commutative diagram of extensions has no trivial factor which partly splits the extension. Moreover since the image of the first arrow is contained in the kernel of the second arrow, the first row of the following commutative diagram splits:

$$\begin{CD}
@. @. 0 @. 0 \\
@. @. @VVV @VVV \\
0 @>>> \operatorname{Ext}^1(G, F_{j'})^* \otimes F_{j'} @>>> \operatorname{Ker}(\alpha) @>>> \operatorname{Ext}^1(G, F_j)^* \otimes F_j @>>> 0 \\
@. @V{=}VV @VVV @VVV \\
0 @>>> \operatorname{Ext}^1(G, F_{j'})^* \otimes F_{j'} @>>> G_{j,j'} @>>> G_j @>>> 0 \\
@. @. @V{\alpha}VV @VVV \\
@. @. G @>{=}>> G \\
@. @. @VVV @VVV \\
@. @. 0 @. 0.
\end{CD}$$

Therefore we have a universal extension:

$$0 \to (\operatorname{Ext}^1(G, F_j)^* \otimes F_j) \oplus (\operatorname{Ext}^1(G, F_{j'})^* \otimes F_{j'}) \to G_{j,j'} \to G \to 0.$$

By repeating the above argument for all j', we obtain the desired universal extension. $\square$

Definition 4.2. We define a filtration of $F^{(n)}$ by $G^p(F^{(n)}) = \mathrm{Ker}(F^{(n)} \to F^{(p-1)})$ for $0 \le p \le n+1$. We have $G^0(F^{(n)}) = F^{(n)}$ and $G^{n+1}(F^{(n)}) = 0$.

Let $\mathrm{End}_G(F^{(n)})$ be the ring of endomorphisms of the object $F^{(n)}$ which preserve the filtration $\{G^p\}$.

Lemma 4.3. *The natural ring homomorphism $\mathrm{End}_G(F^{(n+1)}) \to \mathrm{End}_G(F^{(n)})$ is surjective.*

Proof. Since we only consider endomorphisms preserving the filtration, there is certainly a natural ring homomorphism. For any element $f \in \mathrm{End}_G(F^{(n)})$, we have a commutative diagram in the derived category $D(\mathcal{A})$:

$$\begin{array}{ccccc} F^{(n+1)} & \longrightarrow & F^{(n)} & \longrightarrow & \bigoplus_j \mathrm{Ext}^1(F^{(n)}, F_j)^* \otimes F_j[1] \\ & & {\scriptstyle f}\big\downarrow & & {\scriptstyle f^{**}}\big\downarrow \\ F^{(n+1)} & \longrightarrow & F^{(n)} & \longrightarrow & \bigoplus_j \mathrm{Ext}^1(F^{(n)}, F_j)^* \otimes F_j[1]. \end{array}$$

We obtain a lifting of f to $\mathrm{End}_G(F^{n+1})$ by the axiom of a triangulated category. $\square$

By Lemma 3.6, we have the following

Lemma 4.4. *Let $r_0 = r$ and $r_{m+1} = \sum_j \dim \mathrm{Ext}^1(F^{(m)}, F_j)$ for $m \ge 0$. Then $\dim \mathrm{End}_G(F^{(n)}) = \sum_{m=0}^n r_m$.*

Proof. We have $\dim \mathrm{End}_G(F^{(0)}) = r$. The kernel of the ring homomorphism $\mathrm{End}_G(F^{(m+1)}) \to \mathrm{End}_G(F^{(m)})$ is equal to $\mathrm{Hom}(F^{(m+1)}, \bigoplus_j \mathrm{Ext}^1(F^{(m)}, F_j)^* \otimes F_j)$, whose dimension is equal to $\sum_j \dim \mathrm{Ext}^1(F^{(m)}, F_j) = r_{m+1}$. Therefore we conclude the proof. $\square$

Lemma 4.5. $\dim \mathit{End}(F^{(n)}) \le \sum_{m=0}^n r_m$.

Proof. The assertion follows from Lemma 3.6 and exact sequences. $\square$

Corollary 4.6. *The natural inclusion $\mathrm{End}_G(F^{(n)}) \subseteq \mathrm{End}(F^{(n)})$ is bijective.*

Let $R^{(n)} = \mathrm{End}(F^{(n)})$ and $R^{(n)}_{ij} = \mathrm{Hom}(F^{(n)}_j, F^{(n)}_i)$. Then we can write in a matrix form as $R^{(n)} = (R^{(n)}_{ij})$. Let $M^{(n)} = \mathrm{Ker}(R^{(n)} \to R^{(0)})$ and $M^{(n)}_i = \mathrm{Ker}(R^{(n)} \to R^{(0)} \to k)$, where the last arrow is the projection to the i-th factor.

Proposition 4.7. $R^{(n)} \in (\mathit{Art}_r)$.

Proof. There is a ring homomorphism $R^{(n)} \to R^{(0)} \cong k^r$. The idempotent e_i of $R^{(n)}$ coincides with the projection $F^{(n)} \to F^{(n)}_i \subset F^{(n)}$ to the i-th factor. This gives the k^r-algebra structure of $R^{(n)}$. We know already that $\dim R^{(n)} < \infty$.

We shall prove that $M^{(n)}$ is nilpotent by induction on n. $M^{(0)} = 0$. We assume that $(M^{(n)})^m = 0$ for some $m > 0$, and consider $M^{(n+1)}$. By the assumption, $(M^{(n+1)})^m(F^{(n+1)}) \subset G^{n+1}(F^{(n+1)})$, where $G^{n+1}(F^{(n+1)}) = \bigoplus_j \mathrm{Ext}^1(F^{(n)}, F_j)^* \otimes F_j$. There is an exact sequence

$$0 \to \mathrm{Hom}(F^{(0)}, G^{n+1}(F^{(n+1)})) \to \mathrm{Hom}(F^{(n+1)}, G^{n+1}(F^{(n+1)})) \to \mathrm{Hom}(G^1(F^{(n+1)}), G^{n+1}(F^{(n+1)})).$$

The first homomorphism is bijective, hence the second homomorphism is zero. It follows that $(M^{(n+1)})^m(G^1(F^{(n+1)})) = 0$. Therefore $(M^{(n+1)})^{2m} = 0$. □

Theorem 4.8. *(1) The above constructed $F^{(n)}$ with a natural isomorphism $\phi^{(n)} : R^{(n)}/M^{(n)} \otimes_{R^{(n)}} F^{(n)} \cong F$ is an r-pointed NC deformation of the simple collection $\{F_i\}$ over the ring $R^{(n)}$.*

(2) For any infinitesimal r-pointed deformation (F_R, ϕ) of $\{F_i\}$ over a ring $R \in (Art_r)$, there exist a positive integer n and a ring homomorphism $g : R^{(n)} \to R$ such that $(F_R, \phi) \cong R \otimes_{R^{(n)}} (F^{(n)}, \phi^{(n)})$. Moreover, the induced homomorphism $dg : M^{(n)}/(M^{(n)})^2 \to M_R/M_R^2$ is uniquely determined.

Proof. (1) We only need to show that $F^{(n)}$ is flat over $R^{(n)}$. There are no simple $R^{(n)}$-modules other than the $R^{(n)}/M_i^{(n)}$, and any right $R^{(n)}$-module of finite type is an iterated extension of simple modules. Hence it is sufficient to prove that $\mathrm{Tor}^1_{R^{(n)}}(R^{(n)}/M_i^{(n)}, F^{(n)}) = 0$. By the construction, we have

$$\mathrm{Ker}(\mathrm{End}_G(F^{(k+1)}) \to \mathrm{End}_G(F^{(k)})) \cong \mathrm{Hom}(F, \bigoplus_j \mathrm{Ext}^1(F^{(k)}, F_j)^* \otimes F_j)$$

for $0 \le k < n$. Thus we have exact sequences of $R^{(n)}$-modules

$$0 \to \bigoplus_j \mathrm{Ext}^1(F^{(k)}, F_j)^* \otimes R^{(n)}/M_j^{(n)} \to R^{(k+1)} \to R^{(k)} \to 0.$$

On the other hand, we have sequences of universal extensions

$$0 \to \bigoplus_j \mathrm{Ext}^1(F^{(k)}, F_j)^* \otimes F_j \to F^{(k+1)} \to F^{(k)} \to 0.$$

Since we have $R^{(n)}/M_j^{(n)} \otimes_{R^{(n)}} F^{(n)} \cong F_j$, we conclude that $R^{(k)} \otimes_{R^{(n)}} F^{(n)} \cong F^{(k)}$ and $\mathrm{Tor}^1_{R^{(n)}}(R^{(k)}, F^{(n)}) = 0$ for all k.

(2) The assertion follows because any sequence of iterated non-trivial extensions is dominated by $F^{(n)}$ for a large n. □

Remark 4.9. (1) The above argument gives an explicit construction of the pro-representable hull, i.e., the versal r-pointed NC deformations, for a simple collection in a k-linear abelian category $\mathcal{A}$ as the inverse limit of the $F^{(n)}$.

(2) The presentation of the pro-representable hull by Massey products corresponds to the following exact sequences

$$0 \to \mathrm{Ext}^1(F^{(n+1)}, F_k) \to \bigoplus_j \mathrm{Ext}^1(F^{(n)}, F_j) \otimes \mathrm{Ext}^1(F_j, F_k) \to \mathrm{Ext}^2(F^{(n)}, F_k)$$

where we obtain inductively injective homomorphisms $\mathrm{Ext}^1(F^{(n)}, F) \to (\mathrm{Ext}^1(F, F))^{\otimes(n+1)}$.

(3) The above defined versal family is not universal due to the non-commutativity of the deformation rings. The deformation functor is pro-representable if the following condition is satisfied: for any surjective ring homomorphism $R \to R'$, the natural homomorphism $\mathrm{Aut}_R(F_R) \to \mathrm{Aut}_{R'}(F_{R'})$ for $F_{R'} = R' \otimes_R F_R$ is surjective. Since $\mathrm{Aut}_k(F_R) \cong R$, it follows

that $\mathrm{Aut}_R(F_R)$ coincides with the center $Z(R)$ of R. Since $Z(R) \to Z(R')$ is not necessarily surjective, there is no universal NC deformation of a simple collection in general.

5. r-pointed relative exceptional objects

Exceptional collections yield important examples of semi-orthogonal decompositions. We extend the definition of an exceptional object to a relative version, and prove that it also yields a semi-orthogonal decomposition.

We start with a lemma. We note that, if F_R is an r-pointed NC deformation of some collection over R, then $\mathrm{Hom}(F_R, a)$ has a right R-module structure for any $a \in \mathcal{A}$.

Lemma 5.1. *Let F_R be an r-pointed NC deformation of a simple collection $\{F_i\}$ over $R \in (Art_r)$. Then the following hold.*

(1) $R/M_i \otimes_R F_R \cong F_i$, and $Hom(F_R, F_i) \cong R/M_i$ as right R-modules for all i.

(2) Assume that $\dim Hom(F_i, F_R) = 1$ for all i. Then there exists a permutation σ of r elements such that $Hom(F_i, F_R) \cong R/M_{\sigma(i)}$ as left R-modules for all i.

Proof. (1) Let $f_i \in \mathrm{Hom}(F_R, F_i)$ be the natural projection. Then $M_i = \{r \in R \mid f_i r = 0\}$. Therefore we have our claim.

(2) As left R-modules, we have $\mathrm{Hom}(F_i, F_R) = R/M_j$ for some $j = j(i)$. Then we have $\dim \mathrm{Hom}(F_i, F_{R,k}) = \delta_{jk}$. On the other hand, for each k, there is at least one i such that $\mathrm{Hom}(F_i, F_{R,k}) \neq 0$. Therefore we have a one to one correspondence. □

We denote by $R\mathrm{Hom}(a, b)$ an object $\bigoplus_p \mathrm{Hom}(a, b[p])[-p]$ in $D(k\text{-mod})$.

Definition 5.2. Let $\{F_i\}_{i=1}^r$ be a simple collection in the category of coherent sheaves $(\mathrm{coh}(X))$ on a smooth projective variety, and let $F_R = \bigoplus_i F_{R,i}$ be an r-pointed NC deformation over $R \in (\mathrm{Art}_r)$. The pair (F_R, F) for $F = \bigoplus_i F_i$ is said to be an *r-pointed relative exceptional object* if $R\mathrm{Hom}(F_R, F) \cong R/M$ as right R-modules.

We note that $\mathrm{Hom}(F_{R,i}, F_{R,j})$ may not vanish even though $\mathrm{Hom}(F_i, F_j) = 0$ for $i \neq j$.

Theorem 5.3. *Let (F_R, F) be an r-pointed relative exceptional object over R. Then there is a semi-orthogonal decomposition*

$$D^b(coh(X)) = \langle (\langle F_i \rangle_{i=1}^r)^\perp, \langle F_i \rangle_{i=1}^r \rangle.$$

Proof. Let $G : D^b(\mathrm{coh}(X)) \to D^b(\mathrm{coh}(X))$ be the functor given by a Fourier-Mukai kernel $\mathrm{Cone}(F_R^* \boxtimes_R^{\mathbf{L}} F_R \to \Delta_X)$ on $X \times X$. Then we have distinguished triangles

$$R\mathrm{Hom}(F_R, a) \otimes_R^{\mathbf{L}} F_R \to a \to G(a)$$

for all $a \in D^b(\mathrm{coh}(X))$.

Since $\mathrm{Hom}(F_R, F_R) \cong R$, we obtain $R\mathrm{Hom}(F_R, G(a)) = 0$ by taking $R\mathrm{Hom}(F_R, \bullet)$ of the above triangle. We shall prove that $R\mathrm{Hom}(F_i, G(a)) = 0$ for all i. Let

$$\cdots \to R^{a_k} \to \cdots \to R^{a_1} \to R^{a_0} \to R/M_i \to 0$$

be a free resolution of an R-module R/M_i. Since F_R is flat, we have an exact sequence

$$\cdots \to F_R^{a_k} \to \cdots \to F_R^{a_1} \to F_R^{a_0} \to F_i \to 0.$$

Let $G_k = \mathrm{Im}(F_R^{a_k} \to F_R^{a_{k-1}})$. Then we have $\mathrm{Hom}^p(F_i, G(a)) \cong \mathrm{Hom}^{p-k}(G_k, G(a))$. If we take $k \to \infty$, we conclude that $\mathrm{Hom}^p(F_i, G(a)) = 0$ for any p. Therefore we have our assertion. □

We note that similar statements hold for singular variety X if F_R^* exists in $D^b(\mathrm{coh}(X))$, or more generally for suitable k-linear abelian category and DG enhancement of its derived category.

We consider some examples which yield relative exceptional objects.

Example 5.4. Let X be a singular quadric surface in $\mathbf{P}^3$ defined by an equation $x_1x_2+x_3^2 = 0$.

Let $P = [1 : 0 : 0 : 0] \in X$ be a vertex. Then we have a projection $p : X \setminus \{P\} \to \mathbf{P}^1$. We denote by $\mathcal{O}_X(a)$ the reflexive hull of an invertible sheaf $p^*\mathcal{O}_{\mathbf{P}^1}(a)$ for any integer a. $\mathcal{O}_X(2)$ is an invertible sheaf coming from a hyperplane section in $\mathbf{P}^3$, and we have $\mathcal{O}_X(K_X) \cong \mathcal{O}_X(-4)$. By the vanishing theorem, we have $H^p(X, \mathcal{O}_X(a)) = 0$ for $p > 0$ if $a \ge -3$.

Let $F = \mathcal{O}_X(-1)$. We define an extension $0 \to F \to G \to F \to 0$ by the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X(-1) & \longrightarrow & G & \longrightarrow & \mathcal{O}_X(-1) & \longrightarrow & 0 \\
 & & {\scriptstyle =}\big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(-1) & \longrightarrow & \mathcal{O}_X^2 & \longrightarrow & \mathcal{O}_X(1) & \longrightarrow & 0
\end{array}$$

where the right vertical arrow is obtained from an inclusion $\mathcal{O}_X(-2) \to \mathcal{O}_X$ whose cokernel is supported in the smooth locus. We note that G is a locally free sheaf, hence the extension is non-trivial. Thus the dimension of the local extension at P is $\dim H^0(X, \mathcal{E}xt^1(F, F)) = 1$.

We shall prove that there is no more non-trivial extension, and G is a versal 1-pointed NC deformation of F. Since G is locally free, it is sufficient to prove that $H^1(X, \mathcal{H}om(G, F)) = 0$. We have an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{H}om(G, F) \to \mathcal{O}_X \to \mathcal{E}xt^1(F, F) \to 0$$

Let $H = \mathrm{Ker}(\mathcal{O}_X \to \mathcal{E}xt^1(F, F))$. Then we have $H^1(X, \mathcal{O}_X) = H^1(X, H) = 0$, hence $H^1(X, \mathcal{H}om(G, F)) = 0$.

The base ring of the deformation G is $R = k[t]/(t^2)$, and G is a relative exceptional object over R. Indeed, though X is singular, we have a similar argument as in the above theorem because G is locally free.

Example 5.5. Let X be a singular quadric hypersurface in $\mathbf{P}^4$ defined by an equation $x_1x_2 + x_3x_4 = 0$. It is a cone over $\mathbf{P}^1 \times \mathbf{P}^1$.

Let $P = [1 : 0 : 0 : 0 : 0] \in X$ be a vertex, and $p : X \setminus \{P\} \to \mathbf{P}^1 \times \mathbf{P}^1$ a projection. We denote by $\mathcal{O}_X(a, b)$ the reflexive hull of an invertible sheaf $p^*\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(a, b)$ for any a, b. $\mathcal{O}_X(1, 1)$ is an invertible sheaf coming from a hyperplane section in $\mathbf{P}^4$, and we have $\mathcal{O}_X(K_X) \cong \mathcal{O}_X(-3, -3)$. By the vanishing theorem, we have $H^p(X, \mathcal{O}_X(a, b)) = 0$ for $p > 0$ if $a, b \ge -2$.

Let $F_1 = \mathcal{O}_X(0,-1)$ and $F_2 = \mathcal{O}_X(-1,0)$. We define an extension $0 \to F_2 \to G_1 \to F_1 \to 0$ by the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(-1,0) \arrow[r] \arrow[d, "="'] & G_1 \arrow[r] \arrow[d] & \mathcal{O}_X(0,-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X(-1,0) \arrow[r] & \mathcal{O}_X^2 \arrow[r] & \mathcal{O}_X(1,0) \arrow[r] & 0
\end{tikzcd}$$

where the right vertical arrow is obtained from an inclusion $\mathcal{O}_X(-1,-1) \to \mathcal{O}_X$. We note that G_1 is a locally free sheaf, hence the extension is non-trivial. In a similar way, we construct an extension $0 \to F_1 \to G_2 \to F_2 \to 0$ with G_2 locally free.

We shall prove that $H^1(X, \mathcal{H}om(G_i, F_j)) = 0$ for all i, j. We have an exact sequence

$$0 \to \mathcal{O}_X(-1,1) \to \mathcal{H}om(G_1, F_2) \to \mathcal{O}_X \to \mathcal{E}xt^1(F_1, F_2) \to 0.$$

We have $\dim H^0(X, \mathcal{E}xt^1(F_1, F_2)) = 1$, and $H^1(X, \mathcal{H}om(G_1, F_2)) = 0$ as in the above example. On the other hand, the natural homomorphism $\mathcal{O}_X(1,0) \otimes \mathcal{O}_X(0,-1) \to \mathcal{O}_X(1,-1)$ is surjective. Hence $\mathcal{H}om(G_1, F_1) \to \mathcal{O}_X(1,-1)$ is also surjective, and $H^1(X, \mathcal{H}om(G_1, F_1)) = 0$

$G_1 \oplus G_2$ is a versal 2-pointed NC deformation of $F_1 \oplus F_2$ over

$$R = \begin{pmatrix} k & kt \\ kt & k \end{pmatrix} \mod t^2.$$

$G_1 \oplus G_2$ is a 2-pointed relative exceptional object over R. We note that G_1 and G_2 are exceptional objects, but they do not form an exceptional collection, though there is a semi-orthogonal decomposition with their right orthogonal complement.

Example 5.6. Let $X = \mathbf{P}(1,1,d)$ be the cone over a rational normal curve of degree d. We have reflexive sheaves of rank one $\mathcal{O}_X(a)$ for integers a, and $\mathcal{O}_X(K_X) \cong \mathcal{O}_X(-d-2)$. We consider NC deformations of a sheaf $F = \mathcal{O}_X(-1)$.

Since $\dim H^0(X, \mathcal{O}_X(d-1)) = d$, we have an exact sequence

$$0 \to \mathcal{O}_X(-1)^{d-1} \to \mathcal{O}_X^d \to \mathcal{O}_X(d-1) \to 0.$$

Let $Z \in |\mathcal{O}_X(d)|$ be the smooth curve at infinity. Then we have an exact sequence

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X(d-1) \to \mathcal{O}_Z(d-1) \to 0.$$

Since $\dim H^0(Z, \mathcal{O}_Z(d-1)) = d$, there is a surjective homomorphism $\mathcal{O}_X^d \to \mathcal{O}_Z(d-1)$. Let G be the kernel. Then G is a locally free sheaf of rank d on X. Thus we have the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_X(-1)^{d-1} \arrow[r] \arrow[d, "="'] & G \arrow[r] \arrow[d] & \mathcal{O}_X(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X(-1)^{d-1} \arrow[r] & \mathcal{O}_X^d \arrow[r] \arrow[d] & \mathcal{O}_X(d-1) \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{O}_Z(d-1) \arrow[r, "="] \arrow[d] & \mathcal{O}_Z(d-1) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where all sequences except the first horizontal sequence is exact. Therefore G is a NC deformation of $F = \mathcal{O}_X(-1)$:

$$0 \to \mathcal{O}_X(-1)^{d-1} \to G \to \mathcal{O}_X(-1) \to 0$$

over $R = k[t_1, \dots, t_{d-1}]/(t_1, \dots, t_{d-1})^2$, and $\dim H^0(X, \mathcal{E}xt^1(F, F)) = d - 1$. We have an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{H}om(G, F) \to \mathcal{O}_X^{d-1} \to \mathcal{E}xt^1(F, F) \to 0.$$

Hence $H^1(X, \mathcal{H}om(G, F)) = 0$ as in the above example. Therefore G is a versal NC deformation of $F = \mathcal{O}_X(-1)$.

G is a relative exceptional object over R. $D^b(\mathrm{coh}(X))$ is generated by a relative exceptional collection $(\mathcal{O}_X(-d), G, \mathcal{O}_X)$.

Example 5.7. Let $X = \mathbf{P}(1, 2, 3)$ be a weighted projective surface. X has two singular points P, Q which are Du Val singularities of types A_1, A_2, respectively. We have $\mathcal{O}_X(K_X) \cong \mathcal{O}_X(-6)$. We consider NC deformations of a reflexive sheaf of rank one $F = \mathcal{O}_X(-1)$.

In this example, the non-commutative deformations of F do not terminate after finite steps, though commutative deformations do.

Let us consider local extensions of F at the singular points. At the singular point of type A_1, i.e., a quotient singularity of type $\frac{1}{2}(1, 1)$, it is already known by the previous example that the versal NC local deformations has the base ring $k[s]/(s^2)$.

We can calculate local extensions at a singularity of type A_2, i.e., a quotient singularity of type $\frac{1}{3}(1, 2)$, as follows. A cyclic group $\mathbf{Z}/(3)$ acts on $k[x, y]$, and let $A \subset k[x, y]$ be the invariant subring. We may assume that the sheaf F is represented by the ideal $(x) \cap A \subset A$. There are exact sequences

$$0 \to (x) \cap A \to A \oplus ((y) \cap A) \to (x) \cap A \to 0$$
$$0 \to (x) \cap A \to A^2 \to (y) \cap A \to 0.$$

Therefore the versal NC deformation of the ideal $(x) \cap A$ at Q has the base ring $k[t]/(t^3)$.

We prove that a sequence of iterated non-trivial extensions F_n of F never become locally free for any n by induction on n. If F_n is not locally free at P, then we take an extension

$$0 \to F \to F_{n+1} \to F_n \to 0$$

which induces a non-trivial extension at P and a trivial extension at Q. Then F_{n+1} is not locally free at Q. The same argument works if we interchange P and Q. Therefore we proved our assertion.

Indeed we could prove that the versal NC deformation has a base ring $k\langle s,t\rangle/(s^2,t^3)$, which is infinite dimensional, while its maximal abelian quotient $k[s,t]/(s^2,t^3)$ is finite dimensional.

If we consider 1-pointed NC deformations of reflexive shaves $F_2 = \mathcal{O}_X(-2)$ and $F_3 = \mathcal{O}_X(-3)$, then the results are better. Since F_2 (resp. F_3) is locally free at P (resp. Q), F_2 (resp. F_3) has a versal NC deformation G_2 (resp. G_3) over $k[t]/(t^3)$ (resp. $k[s]/(s^2)$) which is a locally free sheaf of rank 3 (reap. 2). They are relative exceptional objects, and there is a semi-orthogonal decomposition $D^b(\mathrm{coh}(X)) = \langle G_3, G_2, \mathcal{O}_X\rangle$.

We could generalize the above in the following way. Let $X = \mathbf{P}(1,a,b)$ be a weighted projective plane for coprime positive integers a,b with $a<b$. We consider 1-pointed NC deformations of $F_a = \mathcal{O}_X(-a)$ and $F_b = \mathcal{O}_X(-b)$. We could prove that there exist versal deformations G_a and G_b of F_a and F_b, respectively, which are locally free and relative exceptional objects. Moreover there is a semi-orthogonal decomposition $D^b(\mathrm{coh}(X)) = \langle G_b, G_a, \mathcal{O}_X\rangle$.

6. r-pointed relative spherical objects on Calabi-Yau threefolds

We define relative spherical objects after [9] and [1], and prove that a versal multi-pointed NC deformation of a simple collection on a Calabi-Yau 3-fold yields a relative spherical object if the deformation stops infinitesimally and if one more condition holds.

Definition 6.1. Let $\{F_i\}_{i=1}^r$ be a simple collection of coherent sheaves on a smooth projective variety X, and let $F_R = \bigoplus_i F_{R,i}$ be an r-pointed NC deformation of $\{F_i\}$ over $R \in (\mathrm{Art}_r)$. The pair (F_R, F) for $F = \bigoplus_{i=1}^r F_i$ is said to be an *r-pointed relative n-spherical object* over R if the following conditions are satisfied:

(1) There exists a permutation σ of r elements such that $R\mathrm{Hom}(F_R, F_i) \cong R/M_i \oplus R/M_{\sigma(i)}[-n]$ as right R-modules for all i,

(2) $F \otimes \omega_X \cong F$.

More generally, for a triangulated category with a Serre functor S, the second condition can be replaced by $S(F) \cong F[n]$.

Theorem 6.2. *Let (F_R, F) be an r-pointed relative n-spherical object over R. Then there is an autoequivalence T_F of $D^b(coh(X))$, called a* relative spherical twist*, inducing distinguished triangles*

$$RHom(F_R, a) \otimes^{\mathbf{L}}_R F_R \to a \to T_F(a)$$

for all $a \in D^b(coh(X))$.

Proof. ([9]) The functor T_F is given by a Fourier-Mukai kernel $\mathrm{Cone}(F_R^* \boxtimes_R^{\mathbf{L}} F_R \to \Delta_X)$ on $X \times X$.

Since we have $R\mathrm{Hom}(F_R, F_i) \cong R/M_i \oplus R/M_{\sigma(i)}[-n]$, it follows that $T_F(F_i) \cong F_{\sigma(i)}[1-n]$. There is a positive integer m such that $\sigma^m = \mathrm{Id}$. Therefore the homomorphisms $\mathrm{Hom}(F_i, F_j[p]) \to \mathrm{Hom}(F_{\sigma(i)}, F_{\sigma(j)}[p])$ are bijective for all i, j, p.

On the other hand, if $R\mathrm{Hom}(F_i, a) = 0$ for all i, then $T_F(a) \cong a$. Since the F_i and such a's span $D^b(\mathrm{coh}(X))$, we conclude that the functor T_F is fully faithful.

Since $F \otimes \omega_X \cong F$, we have $F_R \otimes \omega_X \cong F_R$. Hence

$$\mathrm{Cone}((F_R^* \otimes \omega_X) \boxtimes_R^{\mathbf{L}} F_R \to \Delta_X \otimes p_1^*\omega_X) \cong \mathrm{Cone}(F_R^* \boxtimes_R^{\mathbf{L}} (F_R \otimes \omega_X) \to \Delta_X \otimes p_2^*\omega_X).$$

It follows that $ST_F \cong T_F S$ for the Serre functor S of $D^b(\mathrm{coh}(X))$, and T_F is an equivalence by [3]. $\square$

Proposition 6.3. *Let (F_R, F) be an r-pointed relative spherical object over R. Then $R^* = Hom_k(R, k)$ is a free right R-module of rank 1.*

Proof. Since $\dim \mathrm{Hom}(F_R, F_i) = \dim \mathrm{Hom}(F_i, F_R) = 1$, we can define $r_i \in R = \mathrm{Hom}(F_R, F_R)$ as a composition of non-zero homomorphisms $F_R \to F_i \to F_R$ up to a constant. Let $\phi \in R^*$ be a homomorphism $R \to k$ such that $\phi(r_i) = 1$ for all i.

We shall prove that ϕ generates R^* as a right R-module. Let $I = \{r \in R \mid \phi r = 0\}$ be the annihilator ideal of ϕ. If $I \ne 0$, then there exist i and $0 \ne r \in I$ such that $M_i r = 0$ for the i-th maximal ideal M_i of R. We know that such non-zero $r \in R$ that $M_i r = 0$ is unique up to a constant for a fixed i, because $\dim \mathrm{Hom}(F_i, F_R) = 1$. It follows that $r = cr_i$ for $0 \ne c \in k$. Then $0 = \phi r(1) = \phi(cr_i) = c$, a contradiction. Hence $I = 0$. $\square$

Theorem 6.4. *Let $\{F_i\}_{i=1}^r$ be a simple collection of coherent sheaves on a smooth projective variety X of dimension 3 such that $F \otimes \omega_X \cong F$ for $F = \bigoplus F_i$. Assume that the versal r-pointed NC deformation F_R is obtained by a finite sequence of iterated non-trivial extensions. Assume moreover that $Hom(F_i, F_R) \ne 0$ for all i. Then (F_R, F) is relatively 3-spherical over R.*

Proof. We have already that $\mathrm{Hom}^1(F_R, F_i) = 0$ for all i. Then we have $\mathrm{Hom}^1(F_R, G) = 0$ for any extension G of the F_i. We have an exact sequence

$$0 \to F_i \to F_R \to G_i \to 0$$

for some G_i for each i. Thus

$$\mathrm{Hom}^1(F_R, G_i) \to \mathrm{Hom}^2(F_R, F_i) \to \mathrm{Hom}^2(F_R, F_R)$$

Since the last terms is dual to $\mathrm{Hom}^1(F_R, F_R) = 0$, we conclude that $\mathrm{Hom}^2(F_R, F_i) = 0$.

Let m_i be the number of appearances of F_i in the iterated extension F_R. Then

$$\begin{aligned}\sum_i m_i &= \dim \mathrm{Hom}(F_R, F_R) = \dim \mathrm{Hom}^3(F_R, F_R) \\ &= \sum_i m_i \dim \mathrm{Hom}^3(F_R, F_i) = \sum_i m_i \dim \mathrm{Hom}(F_i, F_R).\end{aligned}$$

Since $\mathrm{Hom}(F_i, F_R) \ne 0$ for all i, it follows that $\dim \mathrm{Hom}(F_i, F_R) = 1$ for all i. Therefore we have $\mathrm{Hom}^3(F_R, F_i) = 1$ for all i, and we conclude the proof. $\square$

We consider some examples.

Example 6.5. Let $f : X \to Y$ be a projective birational morphism from a smooth variety of dimension 3 to a normal variety such that K_X is relatively trivial and the exceptional locus of f is 1-dimensional. In this case, Y has only terminal Gorenstein singularities. and the irreducible components C_i of the exceptional locus are smooth rational curves.

Let $F_i = \mathcal{O}_{C_i}(-1)$. Then $\{F_i\}$ is a simple collection. The derived dual $F_i^* = RHom(F_i, \mathcal{O}_X)$ is isomorphic to $F_i[-2]$. Indeed, since C is a locally complete intersection, $F_i^*[2]$ is a locally free sheaf on C. Since $Rf_*F_i = 0$, we have $Rf_*F_i^* = 0$ by the duality. Therefore $F_i^* \cong F_i[-2]$.

Let F_R be a versal multi-pointed NC deformation of $F = \bigoplus F_i$. It follows that $F_R^*[2] = RHom(F_R, \mathcal{O}_X)[2]$ is an iterated non-trivial extension of the $\{F_i[-2]\}$ with the reversed order. By the maximality of F_R, we have $F_R^*[2] \cong F_R$. Therefore the additional assumption of the above theorem is satisfied.

We note that there are many simple collections on X. For example, for any disjoint subsets I_j of the set of indexes, if we write $D_j = \bigcup_{i \in I_j} C_i$, then $\{\mathcal{O}_{D_j}\}$ is a simple collection. If l_i is the length of C_i, i.e., the length of the scheme theoretic fiber over a singular point of Y at the generic point of C_i, then a set of fat fibers $\{\mathcal{O}_{k_iC_i}\}$ for some $k_i \le l_i$ is also a simple collection.

Example 6.6. Let $Y \subset k^4$ be a hypersurface of dimension 3 defined by an equation $x_1x_2 + x_3x_4(x_3 + x_4) = 0$. Then there is a resolution of singularities $f : X \to Y$ as in the preceding example. The exceptional locus of f consists of two smooth rational curves C_0, C_1 meeting at a point P transversally. The normal bundles of the C_i are isomorphic to $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$. Let $F_i = \mathcal{O}_{C_i}(-1)$. Then $\{F_0, F_1\}$ is a simple collection.

We can calculate the versal 2-pointed NC deformation $F_R = F_{R,0} \oplus F_{R,1}$ of $\{F_i\}$ as follows. It is given by iterated non-trivial extensions

$$0 \to F_{1-i} \to G_i \to F_i \to 0$$
$$0 \to F_i \to F_{R,i} \to G_i \to 0$$

for $i = 0, 1$, where G_i are invertible sheaves on $C_0 \cup C_1$ of bidegree $(-1, 0)$ and $(0, -1)$ for $i = 0, 1$, respectively. The deformation algebra R has the following form

$$\begin{pmatrix} k + kt^2 & kt \\ kt & k + kt^2 \end{pmatrix} \mod t^3.$$

F_R is a relative 3-spherical object over R:

$$R\mathrm{Hom}(F_R, F_i) \cong R/M_i \oplus R/M_i[-3].$$

Example 6.7. Let $Y \subset k^4$ be a hypersurface of dimension 3 defined by an equation $x_1x_2+x_3^2+x_4^3 = 0$. The blowing up at the origin gives a resolution of singularities $f : X \to Y$ with an exceptional divisor E, which is a quadric cone over $\mathbf{P}^1$ that was considered in

Example 5.4. We use the notation $\mathcal{O}_E(a)$ defined there. We have $K_X = f^*K_Y + E$, $\mathcal{O}_E(E) = \mathcal{O}_E(-2)$ and $K_E = \mathcal{O}_E(-4)$.

Let $e = \mathcal{O}_E(-2)$. Then e is an exceptional object in $D^b(\text{coh}(X))$. Let $\mathcal{D}$ be its left orthogonal complement, and let S be the Serre functor of $\mathcal{D}$. Then $F = \mathcal{O}_E(-1)$ is an object in $\mathcal{D}$. If S' is the Serre functor of $D^b(\text{coh}(X))$, then we have $S'(F) \cong \mathcal{O}_E(-3)[3]$. From an exact sequence

$$0 \to \mathcal{O}_E(-3) \to e^2 \to \mathcal{O}_E(-1) \to 0$$

we deduce that $S(F) \cong F[2]$.

We construct a non-trivial self extension G of F as in Example 5.4. G is a verbsal NC deformation of F over $R = k[t]/(t^2)$, and G is a relative 2-spherical object in $\mathcal{D}$:

$$R\text{Hom}(G, F) \cong R/M \oplus R/M[-2].$$

Remark 6.8. The category $\mathcal{D}$ in the above example was already considered in [6] 4.3. The sheaf G there appeared in [10] 4.13. The construction of tilting generators in [14] can also be considered as a multi-pointed non-commutative deformation of a collection which is not simple. See also [13].

We have a similar example in dimension 4, where we obtain again a relative 2-spherical object:

Example 6.9. Let $Y \subset k^5$ be a hypersurface defined by an equation $x_1x_2 + x_3x_4 + x_5^3 = 0$. The blowing up at the origin gives a resolution of singularities $f : X \to Y$ with an exceptional divisor E, which is a cone over $\mathbf{P}^1 \times \mathbf{P}^1$ that was considered in Example 5.5. We use the notation $\mathcal{O}_E(a, b)$ defined there. We have $K_X = f^*K_Y + 2E$, $\mathcal{O}_E(E) = \mathcal{O}_E(-1, -1)$ and $K_E = \mathcal{O}_E(-3, -3)$.

Let $e_1 = \mathcal{O}_E(-1, -1)$ and $e_2 = \mathcal{O}_E(-2, -2)$. Then (e_2, e_1) is an exceptional collection in $D^b(\text{coh}(X))$. Let $\mathcal{D}$ be its left orthogonal complement, and let S be the Serre functor of $\mathcal{D}$. Then $F_1 = \mathcal{O}_E(-1, 0)$ and $F_2 = \mathcal{O}_E(0, -1)$ are objects in $\mathcal{D}$. If S' is the Serre functor of $D^b(\text{coh}(X))$, then we have $S'(F_1) \cong \mathcal{O}_E(-3, -2)[4]$. From exact sequences

$$0 \to \mathcal{O}_E(-3, -2) \to e_2^2 \to \mathcal{O}_E(-1, -2) \to 0$$

$$0 \to \mathcal{O}_E(-1, -2) \to e_1^2 \to \mathcal{O}_E(-1, 0) \to 0$$

we deduce that $S(F_1) \cong F_1[2]$. Similarly we have $S(F_2) \cong F_2[2]$.

We construct non-trivial self extensions G_1 and G_2 of F_1 and F_2 as in Example 5.5, respectively. Then $G = G_1 \oplus G_2$ is a versal 2-pointed NC deformation of $F = F_1 \oplus F_2$ over $R = \begin{pmatrix} k & kt \\ kt & k \end{pmatrix} \mod t^2$. By the vanishing theorem, we have $H^p(E, \mathcal{O}_E(a, b)) = 0$ for $p > 0$ if $a, b \ge -2$, and G is a relative 2-spherical object in $\mathcal{D}$:

$$R\text{Hom}(G, F_i) \cong R/M_i \oplus R/M_{3-i}[-2].$$

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