Boundedness of Singular Integrals with Flag Kernels on Weighted Flag Hardy Spaces

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BOUNDEDNESS OF SINGULAR INTEGRALS WITH FLAG KERNELS ON WEIGHTED FLAG HARDY SPACES

YONGSHENG HAN, CHIN-CHENG LIN* AND XINFENG WU

Abstract. The main purpose of this paper is to establish the boundedness of singular integrals with flag kernels on weighted Hardy spaces theory associated with flag structures. The theory of weighted Hardy spaces includes weighted Hardy spaces $H^p_{\mathcal{F},w}$, weighted generalized Carleson measure spaces $CMO^p_{\mathcal{F},w}$ (the dual spaces of $H^p_{\mathcal{F},w}$), and the boundedness of singular integrals with flag kernels on these spaces. We also derive a Calderón-Zygmund decomposition and provide interpolation of operators acting on $H^p_{\mathcal{F},w}$. The main tool for our approach is the weighted Littlewood-Paley-Stein theory.

Contents

1. Introduction 1
2. Weighted Hardy spaces, Carleson measure spaces and dual theorem 9
3. Weighted boundedness of singular integrals with flag kernels 27
4. Calderón-Zygmund decomposition and interpolation 35
5. Appendix: Relations among different classes of weights 39
References 41

1. Introduction

The classical Calderón-Zygmund singular integral operator theory is the extension to higher dimensions of the theory of the Hilbert transform. These integral operators have singularity at the origin only, and the nature of this singularity leads to the invariance of these singular integral operators under the classical dilations on $\mathbb{R}^n$ given by $\delta x = (\delta x_1, ..., \delta x_n)$ for $\delta > 0$. On the other hand, the product theory of singular integral operators on $\mathbb{R}^n$ is concerned with those singular integral operators which are invariant

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under the n-fold dilations: $\delta x = (\delta_1 x_1, \delta_2 x_2, ..., \delta_n x_n), \delta_j > 0$ for $1 \le j \le n$. The product theory of $\mathbb{R}^n$ began with the strong maximal function studied by Zygmund, then continued with the Marcinkiewicz multiplier theorem, and more recently has been studied in a variety of directions, for example, product singular integrals and Hardy and BMO spaces studied by Gundy, Chang, R. Fefferman, Journé, Pipher and Stein et al. ([CF1], [Fe], [FS2], [GS], [Jo], [Pi] etc.).

A new extension of product theory came to light with the proof by Müller, Ricci and Stein [MRS] for the L^p boundedness, $1 < p < \infty$, of Marcinkiewicz multipliers on the Heisenberg group $\mathbb{H}^n$. This is surprising since Marcinkiewicz multipliers, which are invariant under a two-parameter group of dilations on $\mathbb{C}^n \times \mathbb{R}$, are bounded on $L^p(\mathbb{H}^n)$, despite the absence of a two-parameter automorphic group of dilation on $\mathbb{H}^n$. Müller, Ricci and Stein proved that the Marcinkiewicz multipliers on the Heisenberg groups are not the classical Calderón-Zygmund singular integrals but are singular integrals with flag kernels. Nagel, Ricci and Stein [NRS] studied a class of operators on nilpotent Lie groups given by the convolution with flag kernels. They proved that product kernels can be written as finite sums of flag kernels and that flag kernels have good regularity, restriction and composition properties. Applying the theory of singular integrals with flag kernels to the study of the $\Box_b$-complex on certain quadratic CR submanifolds of $\mathbb{C}^n$, they obtained L^p regularity for certain derivatives of the relative fundamental solution of $\Box_b$ and for the corresponding Szegö projections onto the null space of $\Box_b$ by showing that the distribution kernels of these operators are finite sums of flag kernels. In order to prove the optimal estimates for solutions of the Kohn-Laplacian for certain classes of model domains in several complex variables, Nagel and Stein [NS] applied a type of singular integral operator whose novel features are related to product theory and flag kernels. These operators differ essentially from the more standard Calderón-Zygmund operators that have been used in these problems hitherto. More recently, Nagel, Ricci, Stein and Wainger [NRSW] (see also [Gł1, Gł2]) further generalized the theory of singular integrals with flag kernels to a more general setting, namely, homogeneous group. For other interesting works in multiparameter harmonic analysis, we refer readers to Hytönen and Martikainen [HM] and Pott and Sehba [PS] and the references therein.

As mentioned, on the Euclidian space convolution with a flag kernel is a special case of product singular integrals. As a consequence, the $L^p, 1 < p < \infty$, boundedness of singular integrals with flag kernels follows automatically from the same result for product singular integrals (see [FS2]). However, since singular integrals with flag kernels have good regularity, a natural question arises: can one develop an appropriate Hardy space theory for singular integrals with flag kernels, which differs from the classical product Hardy space as developed in [GS, CF1, Fe]? Moreover, since the product theory is not available on the Heisenberg groups, it is interesting to ask: can one provide the Hardy space boundedness for the Marcinkiewicz multiplier on the Heisenberg groups [MRS]? To answer these questions, the multiparameter Hardy spaces associated to singular integrals with flag kernels on the Heisenberg groups were developed in [HLS]. An atomic decomposition for flag Hardy spaces was established in [W1]. Very recently, the third author [W2] established the weighted $L^p, 1 < p < \infty$, estimates for singular integrals with flag kernels on homogeneous groups.

The purpose of this paper is to establish the boundedness of singular integrals with flag kernels on weighted Hardy spaces theory associated with with flag structures. The theory of weighted Hardy spaces includes weighted Hardy spaces $H^p_{\mathcal{F},w}$ and weighted generalized Carleson measure spaces $CMO^p_{\mathcal{F},w}$ (the dual spaces of $H^p_{\mathcal{F},w}$). We also derive a Calderón-Zygmund decomposition and provide interpolation of operators acting on $H^p_{\mathcal{F},w}$. To achieve this goal, we will employ the following approaches.

1. The spaces of test functions and distributions are important for developing the Hardy space theory. In the classical case, as in the remarkable work of C. Fefferman and Stein [FS1], these spaces are just the Schwartz test functions and tempered distributions. To study the Hardy space associated flag kernels, in the current paper, we will use the *partial* cancellation conditions to define the test function space. Roughly speaking, any Schwartz test function satisfying the cancellation conditions only in one sub-variable belongs to this space. This condition was used by Nagel, Ricci, Stein and Wainger [NRSW].

2. The classical Calderón reproducing formula was first used by Calderón in [Ca]. Such a reproducing formula is a very powerful tool, in particular, in the theory of wavelet analysis. See [M] for more details. See also (8.17) in [NRSW] for such a formula on homogeneous groups. In this paper, we establish two flag discrete Calderón's reproducing formulae. The first one involves those test functions whose Fourier transforms are compactly supported, and it converges in test function spaces and distributions mentioned above. The second kind of formula is expressed in term of bump functions and it converges only in L^2 norm. Both formulas will be the main tools for developing the whole theory.

3. We establish the Plancherel-Pôlya type inequality. The classical Plancherel-Pôlya inequality says that the L^p norm of f whose Fourier transform has compact support is equivalent to the ℓ^p norm of the restriction of f at appropriate lattices. It was well known that the classical Plancherel-Pôlya inequality plays a crucial role for developing the Littlewood-Paley-Stein theory. See [DHLW] for such inequalities for weighted product Hardy spaces. In this paper, we will establish the Plancherel-Pôlya inequalities associated with the flag structure and provide the Littlewood-Paley-Stein theory. As a consequence, weighted flag Hardy spaces are well defined.

4. We then introduce generalized Carleson measure spaces. It is well known that in the classical one parameter case, the space BMO, as the dual of H^1, can be characterized by the Carleson measure. Moreover, applying atomic decompositions of product Hardy spaces, Chang and R. Fefferman in [CF1] proved that the dual of the product H^1 can be characterized by the product Carleson measure. In this paper, we characterize the dual of weighted flag Hardy spaces via generalized Carleson measure. Our approach is achieved

by applying techniques of weighted sequence spaces, which enables us to avoid using the atomic decompositions.

5. Finally, we establish a Calderón-Zygmund decomposition for $H^p_{\mathcal{F},w}$. The Calderón-Zygmund decomposition played a crucial role in developing Calderón-Zygmund operator theory. This decomposition has many applications in harmonic analysis and PDE's. Such a decomposition for the product Euclidean spaces was first provided by Chang and Fefferman in [CF2] by atomic decompositions. In this paper, Calderón-Zygmund decomposition is achieved by applying the discrete Calderón's reproducing formula. As an application, we derive interpolation results for sublinear operators on $H^p_{\mathcal{F},w}$.

For simplicity, in this paper we shall focus on the case of three parameters, but it is straightforward from the proofs to extend our theory to k $(k>3)$ parameters. To describe the main results in this paper, we first recall the $A_p^{\mathcal{F}}(\mathbb{R}^N)$ weights.

A rectangle R in $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}:=\mathbb{R}^N$ is called a *flag rectangle* if $R=Q_1\times Q_2\times Q_3$, where Q_i's are cubes in $\mathbb{R}^{n_i}$ with side-length

$$\ell(Q_1)\le\ell(Q_2)\le\ell(Q_3).$$

Denote by $\mathcal{R}_{\mathcal{F}}$ the set of flag rectangles associated with $\mathcal{F}$ and by $\mathcal{R}^d_{\mathcal{F}}$ the set of *dyadic* flag rectangles associated with $\mathcal{F}$. For $J=(j_1,j_2,j_3)$, let $\mathcal{R}^J_{\mathcal{F}}$ be the set of dyadic flag rectangles $R=Q_1\times Q_2\times Q_3$ with side-length $\ell(Q_1)=2^{j_1}, \ell(Q_2)=2^{j_1\vee j_2}, \ell(Q_3)=2^{j_1\vee j_2\vee j_3}$, where $a\vee b$ denotes $\max\{a,b\}$. The following *flag maximal function* was introduced in [NRSW]:

$$\mathcal{M}_{\mathcal{F}}(f)(x)=\sup_{\substack{R\ni x\\ R\in\mathcal{R}_{\mathcal{F}}}}\frac{1}{|R|}\int_R|f(y)|dy.$$

The natural class of Muckenhoupt weights associated with $\mathcal{F}$ can be defined as follows.

Definition. Let $1<p<\infty$ and w be a nonnegative locally integrable function on $\mathbb{R}^N$. We say that w is a *flag weight*, denoted by $w\in A_p^{\mathcal{F}}(\mathbb{R}^N)$, if

$$\sup_{R\in\mathcal{R}_{\mathcal{F}}}\Big(\frac{1}{|R|}\int_R w(x)dx\Big)\Big(\frac{1}{|R|}\int_R w(x)^{-1/(p-1)}dx\Big)^{p-1}<\infty.$$

We say that w is in $A_1^{\mathcal{F}}(\mathbb{R}^N)$ if there is a constant C such that

$$\mathcal{M}_{\mathcal{F}}(w)(x)\le Cw(x)\qquad a.e.\ x\in\mathbb{R}^N.$$

Let $A_\infty^{\mathcal{F}}(\mathbb{R}^N):=\bigcup_{1\le p<\infty}A_p^{\mathcal{F}}(\mathbb{R}^N)$. We use $q_w:=\inf\{q:\ w\in A_q^{\mathcal{F}}(\mathbb{R}^N)\}$ to denote the *critical index* of w.

We remark that this class of Muckenhoupt weights is different from the classical weight class $A_p(\mathbb{R}^N)$ or the product weight class $A_p^{\text{pro}}(\mathbb{R}^N)$. Their relations are as follows (see Appendix for details).

$$A_p^{\text{pro}}(\mathbb{R}^N)\subsetneq A_p^{\mathcal{F}}(\mathbb{R}^N)\subsetneq A_p(\mathbb{R}^N)\quad\text{for } 1<p<\infty. \tag{1.1}$$

To develop the weighted Hardy space theory associated with flag singular integrals, as in the classical case, appropriate test functions and distributions are needed. For this purpose, we define three-parameter *flag test functions* as follows.

Definition. A Schwartz function f on $\mathbb{R}^N$ is said to be a *flag test function* in $\mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$ if it satisfies the following partial cancellation conditions

$$\int_{\mathbb{R}^{n_3}} f(x_1,x_2,x_3)x_3^{\alpha}dx_3 = 0 \qquad \text{for all multi-indices } \alpha. \tag{1.2}$$

We would like to point out that these partial cancellation conditions were also considered in [NRSW]. Let $\mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$ denote the dual of $\mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$.

Let $N_1 = n_1+n_2+n_3$, $N_2 = n_2+n_3$ and $N_3 = n_3$. For $i = 1,2,3$, let $\psi^{(i)} \in \mathcal{S}(\mathbb{R}^{N_i})$ satisfy

$$\operatorname{supp}\, \widehat{\psi^{(i)}}(\xi) \subset \{\xi : 1/2 < |\xi| \le 2\} \tag{1.3}$$

and

$$\sum_{j_i\in\mathbb{Z}} \widehat{\psi^{(i)}}(2^{j_i}\xi) = 1 \qquad \text{for all } \xi \in \mathbb{R}^{N_i}\backslash\{0\}. \tag{1.4}$$

The *Littlewood-Paley-Stein square function* of $f \in \mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$ is defined by

$$g_{\mathcal{F}}(f)(x) = \Big(\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^J_{\mathcal{F}}} |\psi_J * f(x_R)|^2\chi_R(x)\Big)^{1/2},$$

where and hereafter x_R denotes the "left-lower corner" of R (i.e. the corner of R with the least value of each coordinate component) and $\psi_J = \widetilde{\psi^{(1)}_{j_1}} * \widetilde{\psi^{(2)}_{j_2}} * \widetilde{\psi^{(3)}_{j_3}}$ with $\widetilde{\psi^{(i)}_{j_i}} = \delta_{\mathbb{R}^{N-N_i}} \otimes \psi^{(i)}_{j_i}, \psi^{(i)}_{j_i}(x) = 2^{-j_iN_i}\psi^{(i)}(2^{-j_i}x), x \in \mathbb{R}^{N_i}, i = 1,2,3.$

Now the weighted Hardy space is defined by the following

Definition. Let $0 < p < \infty$ and $w \in A^{\mathcal{F}}_\infty(\mathbb{R}^N)$. The *weighted flag Hardy space* $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ is defined by

$$H^p_{\mathcal{F},w}(\mathbb{R}^N) = \{f \in \mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N) : g_{\mathcal{F}}(f) \in L^p_w(\mathbb{R}^N)\}$$

with quasi-norm $\|f\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)} := \|g_{\mathcal{F}}(f)\|_{L^p_w(\mathbb{R}^N)}$.

To see that the definition of $H^p_{\mathcal{F},w}$ is independent of the choice of $\{\psi_J\}$, we will prove the following

Theorem 1.1. *Let $0 < p < \infty$ and $w \in A^{\mathcal{F}}_\infty(\mathbb{R}^N)$. Suppose that $\{\psi_J\}, \{\varphi_J\}$ satisfy conditions* (1.3) *and* (1.4)*. Then*

$$\Big\|\Big\{\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^J_{\mathcal{F}}} |\psi_J * f(x_R)|^2\chi_R\Big\}^{1/2}\Big\|_{L^p_w(\mathbb{R}^N)} \approx \Big\|\Big\{\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^J_{\mathcal{F}}} |\varphi_J * f(x_R)|^2\chi_R\Big\}^{1/2}\Big\|_{L^p_w(\mathbb{R}^N)}.$$

Remark 1.1. As mentioned before, it was shown in [NRS] that flag kernels form a subclass of product kernels. Therefore, singular integrals with flag kernels are bounded automatically on the weighted product Hardy spaces $H^p_w(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3})$, when w is a *product* A_∞ *weight* (see [DHLW]). However, by Proposition 5.1, the flag weight is not necessarily a product weight, so our theory of weighted flag Hardy spaces does not fall under the scope of product theory. Moreover, even if $w(x)\equiv 1$, flag Hardy spaces are strictly larger than product Hardy spaces. Nevertheless, we will prove, in Theorem 1.4 below, that singular integrals with flag kernels are bounded on these large spaces $H^p_{\mathcal{F},w}(\mathbb{R}^N)$. This, indeed, was the main motivation to develop the weighted Hardy spaces theory.

Remark 1.2. If $1<p<\infty$ and $w\in A^{\mathcal{F}}_p(\mathbb{R}^N)$, then, by a result in [W2] and a similar argument to the proof of Theorem 1.1, the two spaces $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ and $L^p_w(\mathbb{R}^N)$ coincide with equivalent norms. However, if $p>1$ there is an $w\notin A_p$ such that $H^p_{\mathcal{F},w}\neq L^p_w$. In this regard, we would like to refer the reader to the work of Strömberg and Wheeden [SW1]. Indeed, if $u(x)=|q(x)|^pw(x)$ where $q(x)$ is a polynomial and $w(x)$ satisfies the Muckenhoupt A_p condition, they proved that H^p_u and L^p_u can be identified when all the zeros of $q(x)$ are real and that otherwise H^p_u can be identified with a certain proper subspace of L^p_u. Similar results in product spaces are obtained in [SW2].

To study the dual of $H^p_{\mathcal{F},w}(\mathbb{R}^N)$, we now introduce the following weighted generalized Carleson measure spaces $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$.

Definition. Let $0<p\le 1$, $w\in A^{\mathcal{F}}_\infty(\mathbb{R}^N)$. Suppose that $\{\psi_J\}$ satisfies (1.3) and (1.4). We say that $f\in\mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$ belongs to $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ if

$$\|f\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}:=\sup_{\text{open }\Omega\subset\mathbb{R}^N}\left\{\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{J\in\mathbb{Z}^3}\sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}}\\ R\subset\Omega}}\frac{|R|^2}{w(R)}|\psi_J*f(x_R)|^2\right\}^{1/2}<\infty.$$

Note that the structure associated with a flag is involved in $CMO^p_{\mathcal{F},w}$ spaces. To see that the weighted Carleson measure space $CMO^p_{\mathcal{F},w}$ is well defined, we need the following

Theorem 1.2. *Let $w\in A^{\mathcal{F}}_\infty(\mathbb{R}^N)$. Suppose that $\{\psi_J\},\{\varphi_J\}$ satisfy* (1.3) *and* (1.4). *Then, for $f\in\mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$,*

$$\sup_{\text{open }\Omega\subset\mathbb{R}^N}\left\{\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{J\in\mathbb{Z}^3}\sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}}\\ R\subset\Omega}}\frac{|R|^2}{w(R)}|\psi_J*f(x_R)|^2\right\}^{1/2}$$
$$\approx\sup_{\text{open }\Omega\subset\mathbb{R}^N}\left\{\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{J\in\mathbb{Z}^3}\sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}}\\ R\subset\Omega}}\frac{|R|^2}{w(R)}|\varphi_J*f(x_R)|^2\right\}^{1/2}.$$

The duality between $H^p_{\mathcal{F},w}$ and $CMO^p_{\mathcal{F},w}$ can be stated as follows.

Theorem 1.3. *Let $0<p\le 1$. Then $(H^p_{\mathcal{F},w}(\mathbb{R}^N))^*=CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$. More precisely, if $g\in CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$, the mapping ℓ_g given by $\ell_g(f)=\langle f,g\rangle$, defined initially*

for $f \in \mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$, extends to a continuous linear functional on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ with $\|\ell_g\| \lesssim \|g\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}$.

Conversely, for every $\ell \in (H^p_{\mathcal{F},w}(\mathbb{R}^N))^*$, there exists $g \in CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ so that $\ell = \ell_g$ with $\|g\|_{CMO^p_{\mathcal{F},w}} \lesssim \|\ell\|$.

In order to state the boundedness results for singular integrals with flag kernels on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$, we need to recall some definitions given in [NRS]. Following closely from [NRS], we begin with recalling the definition of a bump function. A k-normalized bump function on $\mathbb{R}^N$ is a C^k function supported on the unit ball with C^k norm bounded by 1. As pointed out in [NRS], the definitions given below are independent of the choices of k, and thus we will simply refer to "normalized bump function" without specifying k.

In this paper, we will consider the singular integrals with the following flag kernels. See [NRSW] for this definition on homogeneous groups.

Definition. A flag kernel is a distribution $\mathcal{K}$ on $\mathbb{R}^N$ which coincides with a C^∞ function away from the coordinate subspace $x_1 = 0$ and satisfies

(i) (differential inequalities) For each $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3$,

$|\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}\mathcal{K}(x)| \lesssim |x_1|^{-n_1-|\alpha_1|}(|x_1|+|x_2|)^{-n_2-|\alpha_2|}(|x_1|+|x_2|+|x_3|)^{-n_3-|\alpha_3|}$

for $x_1 \neq 0$;

(ii) (cancellation conditions)

(a) Given normalized bump functions $\psi_i, i = 1, 2, 3$, on $\mathbb{R}^{n_i}$ and any scaling parameter $r > 0$, define a distribution $\mathcal{K}_{\psi_i,r}$ by setting

$$\langle \mathcal{K}_{\psi_i,r}, \varphi\rangle = \langle \mathcal{K}, (\psi_i)_r \otimes \varphi\rangle \tag{1.5}$$

for any test function $\varphi \in \mathcal{S}(\mathbb{R}^{N-n_i})$. Then the distributions $\mathcal{K}_{\psi_i,r}$ satisfy the differential inequalities

$$|\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}\mathcal{K}_{\psi_1,r}(x_2,x_3)| \lesssim |x_2|^{-n_2-|\alpha_2|}(|x_2|+|x_3|)^{-n_3-|\alpha_3|},$$
$$|\partial^{\alpha_1}_{x_1}\partial^{\alpha_3}_{x_3}\mathcal{K}_{\psi_2,r}(x_1,x_3)| \lesssim |x_1|^{-n_1-|\alpha_1|}(|x_1|+|x_3|)^{-n_3-|\alpha_3|},$$
$$|\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\mathcal{K}_{\psi_3,r}(x_1,x_2)| \lesssim |x_1|^{-n_1-|\alpha_1|}(|x_1|+|x_2|)^{-n_2-|\alpha_2|}.$$

(b) For any bump functions $\bar{\psi}_i$ on $\mathbb{R}^{N-n_i}$ and any parameters $r = (r_1, r_2)$, we define the distributions $\mathcal{K}_{\bar{\psi}_i,r}$ by (1.5). Then the distributions $\mathcal{K}_{\bar{\psi}_i,r}, i = 1, 2, 3$, are one-parameter kernels and satisfy

$$|\partial^{\alpha_i}_{x_i}\mathcal{K}_{\bar{\psi}_i,r}(x_i)| \lesssim |x_i|^{-n_i-|\alpha_i|}.$$

(c) For any bump function ψ on $\mathbb{R}^N$ and $r > 0$, we have

$$|\langle \mathcal{K}, \psi(r\cdot)\rangle| \lesssim 1.$$

Moreover, the corresponding constants that appear in these differential inequalities are independent of r, r_1, r_2 and depend only on α.

A *flag singular integral* $T_{\mathcal{F}}$ is of the form $T_{\mathcal{F}}(f) = \mathcal{K} * f$, where $\mathcal{K}$ is a flag kernel on $\mathbb{R}^N$ defined as above.

A typical example of flag kernel adapted to the flag $\mathcal{F}$, $\{(0,0,0)\} \subset \{(0,0,z)\} \subset \{(0,y,z)\} \subset \mathbb{R}^3$, is

$$\frac{\operatorname{sgn}(y)\operatorname{sgn}(z)}{x\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}$$

(see [NRS]).

The main results in this paper are the following boundedness for flag singular integrals on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ and $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$

Theorem 1.4. *Let $0 < p < \infty$ and $w \in A^{\mathcal{F}}_\infty(\mathbb{R}^N)$. Then the flag singular integral operator $T_{\mathcal{F}}$ is bounded on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$. Moreover, there exists a constant C_p such that*

$$\|T_{\mathcal{F}}(f)\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)} \le C_p \|f\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}.$$

Remark 1.3. As a consequence of Theorem 1.4 and Remark 1.2, we can obtain the boundedness of flag singular integrals on the weighted Lebesgue spaces; that is,

$$\|T_{\mathcal{F}}(f)\|_{L^p_w(\mathbb{R}^N)} \le C_p \|f\|_{L^p_w(\mathbb{R}^N)} \qquad \text{for } 1 < p < \infty \text{ and } w \in A^{\mathcal{F}}_p(\mathbb{R}^N).$$

The following result gives a general principle on the $H^p_{\mathcal{F},w}(\mathbb{R}^N) - L^p_w(\mathbb{R}^N)$ boundedness of operators.

Theorem 1.5. *Suppose $w \in A^{\mathcal{F}}_\infty(\mathbb{R}^N)$ and $0 < p \le 1$. For any sublinear operator T which is bounded on both $L^2(\mathbb{R}^N)$ and $H^p_{\mathcal{F},w}(\mathbb{R}^N)$, then T is bounded from $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ to $L^p_w(\mathbb{R}^N)$. As a consequence, the flag singular integral operator $T_{\mathcal{F}}$ is bounded from $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ to $L^p_w(\mathbb{R}^N)$.*

The $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ boundeness of flag singular integrals is the following

Theorem 1.6. *Let $T_{\mathcal{F}}$ be a singular integral with flag kernel. For $0 < p \le 1$ and $w \in A^{\mathcal{F}}_\infty(\mathbb{R}^N)$, $T_{\mathcal{F}}$ extends uniquely to a bounded operator on $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$. Moreover, there exists a constant C such that*

$$\|T_{\mathcal{F}}(f)\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)} \le C \|f\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}.$$

Note that $CMO^1_{\mathcal{F},w}(\mathbb{R}^N) = BMO_{\mathcal{F},w}(\mathbb{R}^N)$, the dual of $H^1_{\mathcal{F},w}(\mathbb{R}^N)$. Thus, Theorem 1.6 provides the endpoint estimate for singular integrals with flag kernels on $BMO_{\mathcal{F},w}(\mathbb{R}^N)$.

Our last main results are the Calderón-Zygmund decomposition and interpolation for $H^p_{\mathcal{F},w}$.

Theorem 1.7. *Let $w \in A^{\mathcal{F}}_\infty(\mathbb{R}^N)$, $p_1 \in (0,1]$ and $p_1 < p < p_2 < \infty$. Given $f \in H^p_{\mathcal{F},w}(\mathbb{R}^N)$ and $\alpha > 0$, we have the decomposition $f = g + b$, where $g \in H^{p_2}_{\mathcal{F},w}(\mathbb{R}^N)$ and $b \in H^{p_1}_{\mathcal{F},w}(\mathbb{R}^N)$ with $\|g\|^{p_2}_{H^{p_2}_{\mathcal{F},w}(\mathbb{R}^N)} \lesssim \alpha^{p_2-p}\|f\|^p_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}$ and $\|b\|^{p_1}_{H^{p_1}_{\mathcal{F},w}(\mathbb{R}^N)} \lesssim \alpha^{p_1-p}\|f\|^p_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}$.*

We would like to point out that the above result was first proved by Chang and Fefferman in [CF2] for the product Hardy space H^1. As an application of Theorem 1.7, we immediately have the following interpolation of operators.

Theorem 1.8. *Let $w \in A_\infty^{\mathcal{F}}(\mathbb{R}^N)$ and $0 < p_1 < p_2 < \infty$. If T is a sublinear operator bounded from $H^{p_1}_{\mathcal{F},w}(\mathbb{R}^N)$ to $L^{p_1}_w(\mathbb{R}^N)$ and bounded from $H^{p_2}_{\mathcal{F},w}(\mathbb{R}^N)$ to $L^{p_2}_w(\mathbb{R}^N)$, then T is bounded from $H^{p}_{\mathcal{F},w}(\mathbb{R}^N)$ to $L^{p}_w(\mathbb{R}^N)$ for all $p \in (p_1, p_2)$. Similarly, if T is bounded both on $H^{p_1}_{\mathcal{F},w}(\mathbb{R}^N)$ and $H^{p_2}_{\mathcal{F},w}(\mathbb{R}^N)$, then T is bounded on $H^{p}_{\mathcal{F},w}(\mathbb{R}^N)$ for all $p \in (p_1, p_2)$.*

Throughout the paper, for $x = (x_1, x_2, x_3) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}$, let $x^1 = x \in \mathbb{R}^{N_1}$ $x^2 = (x_2, x_3) \in \mathbb{R}^{N_2}$ and $x^3 = x_3 \in \mathbb{R}^{N_3}$. For $J = (j_1, j_2, j_3) \in \mathbb{Z}_3$, we write $j^1 = J \in \mathbb{Z}^3$, $j^2 = (j_2, j_3) \in \mathbb{Z}^2$ and $j^3 = j_3 \in \mathbb{Z}$. Let $a \wedge b = \min\{a, b\}$.

This paper is organized as follows. In the next section, we establish the weighted theory of flag Hardy spaces and Carleson measure spaces. The boundedness of flag singular integrals on these spaces are proved in Section 3. Section 4 is devoted to the Calderón-Zygmund decomposition and interpolation in these spaces. Finally, in section 5, we give some examples/conterexamples to clarify the relationships among the classes of flag weights, classical weights and product weights.

2. Weighted Hardy spaces, Carleson measure spaces and dual theorem

The main purpose of this section is to prove Theorems 1.1, 1.2 and 1.3.

To show Theorem 1.1, we need the following *discrete Calderón reproducing formula.*

Theorem 2.1. *Suppose that $\{\psi_J\}$ satisfy* (1.3) *and* (1.4). *Then*

$$f(x) = \sum_{J\in\mathbb{Z}^3}\sum_{\ell\in\mathbb{Z}^3} 2^{J\cdot n}\psi_J(x - 2^J\ell)\psi_J * f(2^J\ell) = \sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^J_{\mathcal{F}}} |R|\psi_J(x - x_R)\psi_J * f(x_R), \tag{2.1}$$

where $2^J\ell = (2^{j_1}\ell_1, 2^{j_1\vee j_2}\ell_2, 2^{j_1\vee j_2\vee j_3}\ell_3) = x_R$ denotes the left-lower corner of R, $2^{J\cdot n} = 2^{j_1n_1+(j_1\vee j_2)n_2+(j_1\vee j_2\vee j_3)n_3}$ is the measure of $R \in \mathcal{R}^J_{\mathcal{F}}$, and the series converges in $L^2(\mathbb{R}^N)$, $\mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$ and $\mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$.

Proof. The proof of the convergence of the series in L^2 is similar to the classical case. By Fourier transform, $f = \sum_{J\in\mathbb{Z}^3} \psi_J * \psi_J * f$ with the series convergent in $L^2(\mathbb{R}^N)$. Similar to the method used in [FJW], set $g = \psi_J * f$ and $h = \psi_J$. For $\xi \in \mathbb{R}^N$, the Fourier transforms of g and h are respectively given by

$$\widehat{g}(\xi_1,\xi_2,\xi_3) = \widehat{\psi^{(1)}}(2^{j_1}\xi_1, 2^{j_1}\xi_2, 2^{j_1}\xi_3)\widehat{\psi^{(2)}}(2^{j_2}\xi_2, 2^{j_2}\xi_3)\widehat{\psi^{(3)}}(2^{j_3}\xi_3)\widehat{f}(\xi_1,\xi_2,\xi_3),$$
$$\widehat{h}(\xi_1,\xi_2,\xi_3) = \widehat{\psi^{(1)}}(2^{j_1}\xi_1, 2^{j_1}\xi_2, 2^{j_1}\xi_3)\widehat{\psi^{(2)}}(2^{j_2}\xi_2, 2^{j_2}\xi_3)\widehat{\psi^{(3)}}(2^{j_3}\xi_3).$$

Note that the Fourier transforms of g and h are both compactly supported in

$$R_j := \Big\{\xi \in \mathbb{R}^N : |\xi_1| \le 2^{-j_1}\pi,\ |\xi_2| \le 2^{-j_1\vee j_2}\pi, |\xi_3| \le 2^{-j_1\vee j_2\vee j_3}\pi\Big\}.$$

Now we first expand $\widehat{g}$ in a Fourier series on the rectangle R_j

$$\widehat{g}(\xi) = \sum_{\ell\in\mathbb{Z}^3} 2^{J\cdot n}(2\pi)^{-N}\Big(\int_{R_j}\widehat{g}(\xi')e^{i[(2^J\ell)\cdot\xi']}d\xi'\Big)e^{-i[(2^J\ell)\cdot\xi]}$$

and then replace R_J by $\mathbb{R}^N$ since $\widehat{g}$ is supported in R_J. We obtain

$$\widehat{g}(\xi) = \sum_{\ell\in\mathbb{Z}^3} 2^{J\cdot n} g(2^J\ell)e^{-i[(2^J\ell)\cdot\xi]}.$$

Multiplying both sides by $\widehat{h}(\xi)$ and noting $\widehat{h}(\xi)e^{-i[(2^J\ell)\cdot\xi]} = [h(\cdot - 2^J\ell)]^\wedge(\xi)$ yield

$$(g*h)(x) = \sum_{\ell\in\mathbb{Z}^3} 2^{J\cdot n} g(2^J\ell)h(x-2^J\ell).$$

Substituting g by $\psi_J * f$ and h by ψ_J into the above identity gives the discrete Calderón reproducing formula (2.1) and the convergence in $L^2(\mathbb{R}^N)$.

To finish the proof, we only need to show that the series in (2.1) converges in $\mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$; the convergence in $\mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$ then follows from a standard duality argument. The key for doing this is the almost orthogonal estimates: for any $L, M > 0$ and $f \in \mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$,

$$|f*\psi_J(x)| \lesssim 2^{-(|j_1|+|j_2|+|j_3|)L}\frac{1}{(1+|x|)^M}. \tag{2.2}$$

Assume that (2.2) holds for the moment. Then

$$\begin{aligned}
&\Big|\sum_{\ell\in\mathbb{Z}^3} 2^{J\cdot n}(\partial^\alpha\psi_J)(x-2^J\ell)\psi_J * f(2^J\ell)\Big| \\
&\lesssim 2^{-(|j_1|+|j_2|+|j_3|)L'}\sum_{\ell\in\mathbb{Z}^3} 2^{J\cdot n}\frac{1}{(1+|2^{j_1}\ell_1|+|2^{j_1\vee j_2}\ell_2|+|2^{j_1\vee j_2\vee j_3}\ell_3|)^M} \\
&\quad\times\frac{1}{(1+|x_1-2^{j_1}\ell_1|+|x_2-2^{j_1\vee j_2}\ell_2|+|x_3-2^{j_1\vee j_2\vee j_3}\ell_3|)^M} \\
&\lesssim 2^{-(|j_1|+|j_2|+|j_3|)L'}(1+|x|)^{-M} \quad \text{for some } L' > 0,
\end{aligned}$$

which further implies that

$$\sum_{|j_1|,|j_2|,|j_3|>k}\sum_{\ell\in\mathbb{Z}^3} 2^{J\cdot n}\psi_J(x-2^J\ell)\psi_J * f(2^J\ell) \to 0 \qquad \text{in } \mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$$

as $k \to +\infty$.

It remains to verify (2.2). We note that $f(x_1, x_2, \cdot) \in \mathcal{S}_\infty(\mathbb{R}^{n_3})$, the space of Schwartz functions with all moments vanishing, due to $f \in \mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$. Thus for any fixed (x_1, x_2), by the almost orthogonality estimate on $\mathbb{R}^{n_3}$,

$$|\widetilde{\psi}^{(3)}_{j_3} * f(x)| \lesssim 2^{-|j_3|L}(1+|x|)^{-M},$$

which implies

$$|\widetilde{\psi}_J * f(x)| \lesssim 2^{-|j_3|L}2^{(|j_2|+|j_3|)M}(1+|x|)^{-M}. \tag{2.3}$$

Using the fact $f(x_1,\cdot,\cdot) \in \mathcal{S}_\infty(\mathbb{R}^{n_2+n_3})$ and arguing as above, we have

$$|\widetilde{\psi}_J * f(x)| \lesssim 2^{-|j_2|L}2^{(|j_1|+|j_3|)M}(1+|x|)^{-M}. \tag{2.4}$$

We finally use $f \in \mathcal{S}_\infty(\mathbb{R}^N)$ to get

$$|\widetilde{\psi}_J * f(x)| \lesssim 2^{-|j_1|L} 2^{(|j_2|+|j_3|)M}(1+|x|)^{-M}. \tag{2.5}$$

By choosing $L > 100M$ in (2.3) $-$ (2.5) and taking the geometric mean, (2.2) follows. $\square$

The following almost orthogonality estimate will be frequently used in the subsequent part of this section. The proof follows directly from the one-parameter orthogonality estimate (cf. [HLLL, page 2840]). See also [NRSW] for similar estimates on homogeneous groups.

Lemma 2.2. *Given positive integers L and M, there exists a constant $C = C(L, M) > 0$ such that*

$$|\psi_J * \varphi_{J'}(x)| \le C 2^{-(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)L} \prod_{i=1}^{3} \frac{\max\limits_{1\le k\le i} 2^{(j_k \vee j_k')M}}{\Big(\max\limits_{1\le k\le i} 2^{j_k\vee j_k'} + |x_i|\Big)^{n_i+M}}, \tag{2.6}$$

where $\{\psi_J\}$ and $\{\varphi_{J'}\}$ satisfy (1.3).

Remark 2.1. The above Lemma 2.2 also holds if the functions $\{\psi^{(i)}\}$ and $\{\varphi^{(i)}\}$, $i = 1, 2, 3$, satisfy moment conditions up to order M_0 (see Theorem 3.3 for choosing such an M_0):

$$\int_{\mathbb{R}^{N_i}} \psi^{(i)}(x^i)(x^i)^{\alpha_i} dx^i = \int_{\mathbb{R}^{N_i}} \varphi^{(i)}(y^i)(y^i)^{\beta_i} dy^i = 0 \qquad \text{for all multi-indices } |\alpha_i|, |\beta_i| \le M_0.$$

In such a case, almost orthogonality estimates hold for all positive integers M and $L \le M_0 + 1$.

The following the maximal function estimate is also frequently needed.

Lemma 2.3. *Let $J \in \mathbb{Z}^3$, $R = Q_1 \times Q_2 \times Q_3 \in \mathcal{R}_{\mathcal{F}}^J$ and $M \ge 2N$. Then, for any $x, \bar{x} \in R$ and $\delta \in (\frac{N}{N+M}, 1]$, we have*

$$\begin{aligned} &\sum_{R' \in \mathcal{R}_{\mathcal{F}}^{J'}} |R'| \Big[\prod_{i=1}^{3} \frac{\max\limits_{1\le k\le i} 2^{(j_k\vee j_k')M}}{\Big(\max\limits_{1\le k\le i} 2^{(j_k\vee j_k')} + |\bar{x}_i - x_i'|\Big)^{n_i+M}} \Big] |g(x')| \\ &\quad \le C_N \Big\{ \prod_{i=1}^{3} [2^{3n_i(j_i - j_i')} \vee 1] \Big\}^{\frac{1}{\delta}-1} \Big\{ \mathcal{M}_{\mathcal{F}} \Big[\Big(\sum_{R' \in \mathcal{R}_{\mathcal{F}}^{J'}} |g(x')|^2 \chi_{R'} \Big)^{\frac{\delta}{2}} \Big](x) \Big\}^{\frac{1}{\delta}}, \ \forall\, x' \in R', \end{aligned} \tag{2.7}$$

where C_N is a constant depending only on N.

Proof. The proof of this lemma is similar to the classical case. For $i = 1, 2, 3$ and $r_i \in \mathbb{Z}_+$, set

$$A_0^i = \Big\{ Q_i' : |\bar{x}_i - x_i'| \le \max_{1\le k\le i} 2^{(j_k\vee j_k')} \Big\}$$

and

$$A_{r_i}^i = \Big\{ Q_i' : 2^{r_i - 1} < \frac{|\bar{x}_i - x_i'|}{\max\limits_{1\le k\le i} 2^{(j_k\vee j_k')}} \le 2^{r_i} \Big\}.$$

For any fixed $r = (r_1, r_2, r_3)$ with each $r_i \geq 0$, write

$$E_r = \Big\{(w_1, w_2, w_3) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3} : |w_i - \bar{x}_i| \leq \max_{1\leq k\leq i}\{2^{r_i} 2^{(j_k \vee j'_k)+1}\},\ i = 1, 2, 3\Big\}.$$

Then $R \subset E_r$ and, for each $R' \in \mathscr{A}_r := A^1_{r_1} \times A^2_{r_2} \times A^3_{r_3}$, $R' \subset E_r$. Obviously, $E_r \in \mathcal{R}_{\mathcal{F}}$ and $|E_r| \leq C \prod_{i=1}^3 \max_{1\leq k\leq i} 2^{n_i\{r_i+(j_k\vee j'_k)\}}$.

For $\frac{N}{N+M} < \delta \leq 1$ and for any $x \in R$, (2.7) is majorized by

$$\begin{aligned}
&\sum_{r\in\mathbb{N}^3}\Big[\prod_{i=1}^3 2^{-r_i(M+n_i)}(\max_{1\leq k\leq i} 2^{-n_i(j_k\vee j'_k)})\Big]|R'|\Big(\sum_{R'\in\mathscr{A}_r}|g(x')|^\delta\Big)^{\frac{1}{\delta}}\\
&= \sum_{r\in\mathbb{N}^3}\Big[\prod_{i=1}^3 2^{-r_i(M+n_i)}(\max_{1\leq k\leq i} 2^{-n_i(j_k\vee j'_k)})\Big]|R'|^{1-\frac{1}{\delta}}|E_r|^{\frac{1}{\delta}}\Big(\frac{1}{|E_r|}\int_{E_r}\sum_{R'\in\mathscr{A}_r}|g(x')|^\delta\chi_{R'}(y)dy\Big)^{\frac{1}{\delta}}\\
&\leq \Big(\sum_{r\in\mathbb{N}^3}\prod_{i=1}^3 2^{-r_i(M-2n_i)}\Big)\Big(\prod_{i=1}^3[2^{3n_i(j_i-j'_i)}\vee 1]\Big)^{1/\delta-1}\Big(\mathcal{M}_{\mathcal{F}}(\sum_{R'\in\mathcal{R}^{J'}_{\mathcal{F}}}|g(x')|^\delta\chi_{R'})(x)\Big)^{\frac{1}{\delta}}.
\end{aligned}$$

Since $M > 2N$, the last term is then bounded by

$$C\Big(\prod_{i=1}^3[2^{3n_i(j_i-j'_i)}\vee 1]\Big)^{1/\delta-1}\Big(\mathcal{M}_{\mathcal{F}}(\sum_{R'\in\mathcal{R}^{J'}_{\mathcal{F}}}|g(x')|^\delta\chi_{R'})(x)\Big)^{1/\delta}.$$

This proves Lemma 2.3. $\square$

For $i = 1, 2, 3$, write $x = (\bar{x^i}, x^i) \in \mathbb{R}^{N-N_i} \times \mathbb{R}^{N_i}$. We say that $w \in A^{(i)}_p(\mathbb{R}^N)$ if $w(\bar{x^i}, \cdot)$ is a classical $A_p(\mathbb{R}^{N_i})$ weight uniformly in $\bar{x^i}$; that is,

$$\sup_{\substack{Q\subset R^{N_i}\\ \bar{x^i}\in\mathbb{R}^{N-N_i}}}\left(\frac{1}{|Q|}\int_Q w(\bar{x^i}, x^i)dx^i\right)\left(\frac{1}{|Q|}\int_Q w(\bar{x^i}, x^i)^{-1/(p-1)}dx^i\right)^{p-1} < \infty.$$

Let $\mathcal{M}_i$ denote the Hardy-Littlewood maximal operator on $\mathbb{R}^{N_i}$. The *lifted maximal operator* $\widetilde{\mathcal{M}_i}$ on $\mathbb{R}^N$ was introduced in [NRSW] by

$$\widetilde{\mathcal{M}_i} := \delta_{\mathbb{R}^{N-N_i}} \otimes \mathcal{M}_i,$$

where $\delta_{\mathbb{R}^{N-N_i}}$ is the Dirac mass at $\mathbf{0} \in \mathbb{R}^{N-N_i}$.

The following result was proved in [W2].

Lemma 2.4. *Let $1 < p < \infty$ and w be a nonnegative measurable function on $\mathbb{R}^N$. The following statements are equivalent:*

(i) $w \in A^{\mathcal{F}}_p(\mathbb{R}^N)$;
(ii) $w \in A^{(1)}_p \cap A^{(2)}_p \cap A^{(3)}_p(\mathbb{R}^N)$;
(iii) $\widetilde{\mathcal{M}_3} \circ \widetilde{\mathcal{M}_2} \circ \widetilde{\mathcal{M}_1}$ *is bounded on* $L^p_w(\mathbb{R}^N)$;
(iv) $\mathcal{M}_{\mathcal{F}}$ *is bounded on* $L^p_w(\mathbb{R}^N)$.

Using Lemma 2.4 and applying Rubio de Francias's extropolation (or the argument in [AJ]), one can easily obtain the following weighted Fefferman-Stein vector-valued inequality.

Corollary 2.5. *Let $w \in A_p^{\mathcal{F}}(\mathbb{R}^N)$ and $\{f_j\}_{j\in\mathbb{Z}} \in L_w^p(\ell^q)$. Then, for all $1 < p, q < \infty$,*

$$\int_{\mathbb{R}^N} |\{\mathcal{M}_{\mathcal{F}}(\{f_j\})(x)\}|_{\ell^q}^p w(x)dx \le C \int_{\mathbb{R}^N} |\{f_j(x)\}|_{\ell^q}^p w(x)dx,$$

where $|\cdot|_{\ell^q}$ means the classical ℓ^q-norm.

We now are ready to show Theorem 1.1.

Proof of Theorem 1.1. Let $f \in \mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$ and $w \in A_\infty^{\mathcal{F}}(\mathbb{R}^N)$. We denote $x_R = 2^J\ell$ and $x_{R'} = 2^{j'}\ell'$. Applying Theorem 2.1, Lemma 2.2 with $M > N[((q_w/p) - 1) \vee 2]$ and $L = 10M$ and Lemma 2.3, we get that, for $\frac{N}{N+M} < \delta < (\frac{p}{q_w} \wedge 1)$ and for any $x \in R$,

$$\begin{aligned}
|(\psi_J * f)(x_R)| &\approx \Big| \sum_{J'\in\mathbb{Z}^3} \sum_{R'\in\mathcal{R}_{\mathcal{F}}^{J'}} |R'|\psi_J * \varphi_{J'}(x_R - x_{R'})\varphi_J * f(x_{R'}) \Big| \\
&\lesssim \sum_{J'\in\mathbb{Z}^3} 2^{-(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)L} \\
&\quad \times \sum_{R'\in\mathcal{R}_{\mathcal{F}}^{J'}} |R'| \Big[\prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{(j_k\vee j_k')M}}{(\max_{1\le k\le i} 2^{(j_k\vee j_k')} + |x_{Q_i} - x_{Q_i'}|)^{n_i+M}} \Big] |\varphi_{J'} * f(x_{R'})| \\
&\lesssim \sum_{J'\in\mathbb{Z}^3} 2^{-(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)L'} \Big\{ \mathcal{M}_{\mathcal{F}}\Big[\Big(\sum_{R'\in\mathcal{R}_{\mathcal{F}}^{J'}} |\varphi_{J'} * f(x_{R'})|^2 \chi_{R'} \Big)^{\frac{\delta}{2}} \Big](x) \Big\}^{\frac{1}{\delta}},
\end{aligned}$$

where $L' = L - 3N(1/\delta - 1) > 7M > 0$.

Squaring both sides, then multiplying χ_R, summing over all $J \in \mathbb{Z}^3$ and $R \in \mathcal{R}_{\mathcal{F}}^J$, and finally applying Hölder's inequality, we obtain that, for all $x \in \mathbb{R}^N$ and $\frac{N}{N+M} < \delta < (\frac{p}{q_w} \wedge 1)$,

$$\begin{aligned}
&\sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^J} |\psi_J * f(x_R)|^2 \chi_R \\
&\lesssim \sum_{J\in\mathbb{Z}^3} \Big\{ \sum_{J'\in\mathbb{Z}^3} 2^{-(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)L'} \Big\} \Big\{ \sum_{J'\in\mathbb{Z}^3} 2^{-(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)L'} \\
&\quad \times \Big\{ \mathcal{M}_{\mathcal{F}}\Big[\Big(\sum_{R'\in\mathcal{R}_{\mathcal{F}}^{J'}} |\varphi_{J'} * f(x_{R'})|^2 \chi_{R'} \Big)^{\frac{\delta}{2}} \Big](x) \Big\}^{\frac{2}{\delta}} \Big\} \\
&\lesssim \sum_{J'\in\mathbb{Z}^3} \Big\{ \mathcal{M}_{\mathcal{F}}\Big[\Big(\sum_{R'\in\mathcal{R}_{\mathcal{F}}^{J'}} |\varphi_{J'} * f(x_{R'})|^2 \chi_{R'} \Big)^{\frac{\delta}{2}} \Big](x) \Big\}^{\frac{2}{\delta}},
\end{aligned}$$

where we used the estimates

$$\sum_{J'\in\mathbb{Z}^3} 2^{-(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)L'} \le C \qquad \text{and} \qquad \sum_{J\in\mathbb{Z}^3} 2^{-(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)L'} \le C$$

in the last inequality. Note that $(2\wedge p)/\delta > q_w$ implies $w \in A^{\mathcal{F}}_{p/\delta}(\mathbb{R}^N)$. Applying Corollary 2.5 with $L^{p/\delta}_w(\ell^{2/\delta})$ yields

$$\Big\|\Big\{\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^J_{\mathcal{F}}}|\psi_J * f(x_R)|^2\chi_R\Big\}^{1/2}\Big\|_{L^p_w(\mathbb{R}^N)} \lesssim \Big\|\Big\{\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^J_{\mathcal{F}}}|\varphi_J * f(x_R)|^2\chi_R\Big\}^{1/2}\Big\|_{L^p_w(\mathbb{R}^N)}.$$

The converse inequality follows by symmetry. □

As a consequence of Theorem 1.1, we obtain a density result of $H^p_{\mathcal{F},w}$ which will be useful to show the $H^p_{\mathcal{F},w} - L^p_w$ boundedness of operators, the weak density of $CMO^p_{\mathcal{F},w}$ and the Calderón-Zygmund decomposition for $H^p_{\mathcal{F},w}$.

Corollary 2.6. *Let $0 < p < \infty$ and $w \in A^{\mathcal{F}}_\infty(\mathbb{R}^N)$. Then $\mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$ is dense in $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ and, in consequence, $L^2(\mathbb{R}^N)\cap H^p_{\mathcal{F},w}(\mathbb{R}^N)$ is dense in $H^p_{\mathcal{F},w}(\mathbb{R}^N)$.*

Proof. Let $f \in H^p_{\mathcal{F},w}(\mathbb{R}^N)$. For any fixed $L > 0$, denote

$$E_L = \{(J,R) : |j_1|,|j_2|,|j_3| \le L, R \subset B(0,L)\}$$

and

$$f_L(x) = \sum_{(J,R)\in E_L}|R|\psi_J(x-x_R)\psi_J * f(x_R).$$

Since $\psi_J \in \mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$, it is obvious that $f_L \in \mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$. Repeating the proof of Theorem 1.1, we conclude that $\|f_L\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)} \lesssim \|f\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}$.

To see that f_L tends to f in $H^p_{\mathcal{F},w}(\mathbb{R}^N)$, we use the discrete Calderón reproducing formula to write

$$[g_{\mathcal{F}}(f-f_L)(x)]^2 = \sum_{J'\in\mathbb{Z}^3}\sum_{R'\in\mathcal{R}^{J'}_{\mathcal{F}}}\Big|\sum_{(J,R)\in E^c_L}|R|\psi_{J'} * \psi_J(x_{R'}-x_R)\psi_J * f(x_R)\Big|^2\chi_{R'}(x).$$

Now repeating the same argument as in the proof of Theorem 1.1 again, we get

$$\|g_{\mathcal{F}}(f-f_L)\|_{L^p_w(\mathbb{R}^N)} \lesssim \Big\|\Big\{\sum_{(J,R)\in E^c_L}|\psi_J * f(x_R)|^2\chi_R\Big\}^{1/2}\Big\|_{L^p_w(\mathbb{R}^N)},$$

where the last term tends to 0 as L goes to infinity. This implies that f_L tends to f in $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ norm and hence the proof is finished. □

We follow the classical case (see [St, GR]) to get the following lemma.

Lemma 2.7. *Suppose $w \in A^{\mathcal{F}}_\infty(\mathbb{R}^n)$ and $q > q_w$. There exist $0 < \delta < 1 < q < \infty$ such that, for all flag rectangles R and all measurable subsets A of R,*

$$\Big(\frac{|A|}{|R|}\Big)^q \lesssim \frac{w(A)}{w(R)} \lesssim \Big(\frac{|A|}{|R|}\Big)^\delta.$$

In particular, the measure $w(x)dx$ is doubling with respect to flag rectangles.

Lemma 2.8. *Let $w \in A_\infty^{\mathcal{F}}(\mathbb{R}^N)$. Then, for all flag rectangles R and R' and for $q > q_w$,*

$$\frac{w(R')}{w(R)} \lesssim \prod_{i=1}^{3} \left(\frac{|Q_i|}{|Q_i'|} \vee \frac{|Q_i'|}{|Q_i|} \right)^q \left(1 + \frac{|x_{Q_i} - x_{Q_i'}|}{\ell(Q_i) \vee \ell(Q_i')} \right)^{n_i q}.$$

Proof. Observe that $Q_i' \subset A_i Q_i,\ i = 1, 2, 3$, where $A_i = C(\ell(Q_i) \vee \ell(Q_i') + |x_{Q_i} - x_{Q_i'}|)/\ell(Q_i)$. This implies $R' \subset \bar{R}$, where $\bar{R} = C[(A_1Q_1) \times (A_2Q_2) \times (A_3Q_3)]$. By Lemma 2.7, for any $q > q_w$,

$$\begin{aligned} \frac{w(R')}{w(R)} \le \frac{w(\bar{R})}{w(R)} &\lesssim \left[\frac{|\bar{R}|}{|R|} \right]^q \lesssim \prod_{i=1}^{3} \left[\frac{\ell(Q_i) \vee \ell(Q_i') + |x_{Q_i} - x_{Q_i'}|}{\ell(Q_i)} \right]^{n_i q} \\ &\lesssim \prod_{i=1}^{3} \left[\frac{|Q_i|}{|Q_i'|} \vee \frac{|Q_i'|}{|Q_i|} \right]^q \left[1 + \frac{|x_{Q_i} - x_{Q_i'}|}{\ell(Q_i) \vee \ell(Q_i')} \right]^{n_i q}. \end{aligned}$$

Hence the proof is concluded. □

We now show Theorem 1.2.

Proof of Theorem 1.2. For $R \in \mathcal{R}_{\mathcal{F}}^J$ and $R' \in \mathcal{R}_{\mathcal{F}}^{J'}$, set

$$S_R = |\psi_J * f(x_R)|^2 \qquad \text{and} \qquad T_{R'} = |\varphi_{J'} * f(x_{R'})|^2.$$

Theorem 2.1 and Lemma 2.2 yield

$$\begin{aligned} S_R^{\frac{1}{2}} &= \Big| \sum_{J' \in \mathbb{Z}^3} \sum_{R' \in \mathcal{R}_{\mathcal{F}}^{J'}} |R'| \varphi_{J'} * f(x_{R'}) \psi_J * \varphi_{J'}(x_R - x_{R'}) \Big| \\ &\lesssim \sum_{J' \in \mathbb{Z}^3} \sum_{R' \in \mathcal{R}_{\mathcal{F}}^{J'}} 2^{-(|j_1 - j_1'| + |j_2 - j_2'| + |j_3 - j_3'|)NL} |R'| \\ &\quad \times \Big[\prod_{i=1}^{3} \frac{\max\limits_{1 \le k \le i} 2^{(j_k \vee j_k')M}}{(\max\limits_{1 \le k \le i} 2^{(j_k \vee j_k')} + |x_{Q_i} - x_{Q_i'}|)^{n_i + M}} \Big] |\varphi_{J'} * f(x_{R'})| \\ &\lesssim \sum_{J' \in \mathbb{Z}^3} \sum_{R' \in \mathcal{R}_{\mathcal{F}}^{J'}} r(R, R') P(R, R') T_{R'}^{\frac{1}{2}}, \end{aligned}$$

where

$$r(R, R') := \prod_{i=1}^{3} \left[\frac{|Q_i|}{|Q_i'|} \wedge \frac{|Q_i'|}{|Q_i|} \right]^L \quad \text{and} \quad P(R, R') := \prod_{i=1}^{3} \frac{1}{\left(1 + \frac{|x_{Q_i} - x_{Q_i'}|}{\max_{1 \le k \le i} 2^{(j_k \vee j_k')}} \right)^{n_i + M}}.$$

Squaring both sides, multiplying by $|R|^2/w(R)$, adding up all the terms over $J \in \mathbb{Z}^3$, $R \in \mathcal{R}_{\mathcal{F}}^J, R \subset \Omega$ and applying Hölder's inequality, we obtain

$$\sup_{\Omega} \Big\{ \frac{1}{[w(\Omega)]^{2/p-1}} \sum_{\substack{R \in \mathcal{R}_{\mathcal{F}}^d \\ R \subset \Omega}} |R|^2 w(R)^{-1} S_R \Big\}$$

$$\lesssim \sup_{\Omega}\Big\{\frac{1}{[w(\Omega)]^{2/p-1}}\sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}}\\ R\subset\Omega}}|R|^2w(R)^{-1}\Big[\sum_{R'\in\mathcal{R}^d_{\mathcal{F}}}r(R,R')P(R,R')\Big]$$
$$\times\Big[\sum_{R'\in\mathcal{R}^d_{\mathcal{F}}}r(R,R')P(R,R')T_{R'}\Big]\Big\}$$
$$\lesssim \sup_{\Omega}\Big\{\frac{1}{[w(\Omega)]^{2/p-1}}\sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}}\\ R\subset\Omega}}\sum_{R'\in\mathcal{R}^d_{\mathcal{F}}}|R|^2w(R)^{-1}r(R,R')P(R,R'))T_{R'}\Big\}.$$

Here and hereafter, we use $\sum_{R\in\mathcal{R}^d_{\mathcal{F}}}$ to denote $\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^J_{\mathcal{F}}}$ and similarly for $\sum_{R'\in\mathcal{R}^d_{\mathcal{F}}}$. Applying Lemma 2.8, we get

$$\tag{2.8}\begin{aligned}&\sup_{\Omega}\Big\{\frac{1}{[w(\Omega)]^{2/p-1}}\sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}}\\ R\subset\Omega}}|R|^2w(R)^{-1}S_R\Big\}\\ &\quad\lesssim \sup_{\Omega}\Big\{\frac{1}{[w(\Omega)]^{2/p-1}}\sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}}\\ R\subset\Omega}}\sum_{R'\in\mathcal{R}^d_{\mathcal{F}}}|R'|^2w(R')^{-1}\widetilde{r}(R,R')\widetilde{P}(R,R'))T_{R'}\Big\}.\end{aligned}$$

Here the definition of $\widetilde{r}(R,R')$ and $\widetilde{P}(R,R')$ are defined as $r(R,R')$ and $P(R,R')$ with smaller L and M. Since L and M can be chosen arbitrarily large, in what follows, we still use $r(R,R')$ and $P(R,R')$ to denote $\widetilde{r}(R,R')$ and $\widetilde{P}(R,R')$, respectively.

To finish the proof, it suffices to show that the right hand side of (2.8) is bounded by

$$C\sup_{\overline{\Omega}}\Big\{\frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}}\sum_{\substack{R'\in\mathcal{R}^d_{\mathcal{F}}\\ R'\subset\overline{\Omega}}}|R'|^2w(R')^{-1}T_{R'}\Big\}.$$

We point out that $r(R,R')$ and $P(R,R')$ characterize the geometrical properties between two flag rectangles R and R'. Namely, when the difference of the sizes of R and R' grows bigger, $r(R,R')$ becomes smaller; when the distance between R and R' gets larger, $P(R,R')$ becomes smaller. The following argument is quite geometric. To be precise, we shall first decompose the set of dyadic flag rectangles $\{R'\}$ into annuli according to the distance of R and R'. Next, in each annuli, precise estimates are given by considering the difference of the sizes of R and R'. Finally, add up all the estimates in each annuli to finish the proof.

We now turn to details. For $J=(j_1,j_2,j_3)\in\mathbb{Z}^3$ and $R\in\mathcal{R}^d_{\mathcal{F}}$, denote

$$R_J=R_{j_1,j_2,j_3}=(2^{j_1}Q_1)\times(2^{j_1\vee j_2}Q_2)\times(2^{j_1\vee j_2\vee j_3}Q_3),\quad \Omega^J=\Omega^{j_1,j_2,j_3}=\bigcup_{R\subset\Omega}3R_J.$$

For any flag rectangle $R\subset\Omega$ and $J=(j_1,j_2,j_3)\in\mathbb{Z}^3_+$, let

$$\mathscr{A}_{0,0,0}(R)=\{R':3R'_{0,0,0}\cap 3R\neq\varnothing\},$$
$$\mathscr{A}_{j_1,0,0}(R)=\{R':3R'_{j_1,0,0}\cap 3R\neq\varnothing \text{ and } 3R'_{j_1-1,0,0}\cap 3R=\varnothing\},$$
$$\mathscr{A}_{0,j_2,0}(R)=\{R':3R'_{0,j_2,0}\cap 3R\neq\varnothing \text{ and } 3R'_{0,j_2-1,0}\cap 3R=\varnothing\},$$

$$
\begin{aligned}
\mathscr{A}_{0,0,j_3}(R) &= \{R' : 3R'_{0,0,j_3} \cap 3R \neq \varnothing \text{ and } R'_{0,0,j_3-1} \cap 3R = \varnothing\},\\
\mathscr{A}_{j_1,j_2,0}(R) &= \{R' : 3R'_{j_1,j_2,0} \cap 3R \neq \varnothing,\ 3R'_{j_1-1,j_2,0} \cap 3R = \varnothing \text{ and } 3R'_{j_1,j_2-1,0} \cap 3R = \varnothing\},\\
\mathscr{A}_{j_1,0,j_3}(R) &= \{R' : 3R'_{j_1,0,j_3} \cap 3R \neq \varnothing,\ 3R'_{j_1-1,0,j_3} \cap 3R = \varnothing \text{ and } 3R'_{j_1,0,j_3-1} \cap 3R = \varnothing\},\\
\mathscr{A}_{0,j_2,j_3}(R) &= \{R' : 3R'_{0,j_2,j_3} \cap 3R \neq \varnothing,\ 3R'_{0,j_2-1,j_3} \cap 3R = \varnothing \text{ and } 3R'_{0,j_2,j_3-1} \cap 3R = \varnothing\},\\
\mathscr{A}_{j_1,j_2,j_3}(R) &= \{R' : 3R'_{j_1,j_2,j_3} \cap 3R \neq \varnothing,\ 3R'_{j_1-1,j_2,j_3} \cap 3R = \varnothing,\\
&\qquad 3R'_{j_1,j_2-1,j_3} \cap 3R = \varnothing \text{ and } 3R'_{j_1,j_2,j_3-1} \cap 3R = \varnothing\}.
\end{aligned}
$$

Given a dyadic flag rectangle $R \subset \Omega$, for each flag rectangle R', there exist $J \in \mathbb{N}^3$ such that $R' \in \mathscr{A}_J(R)$ and thus $\mathcal{R}^d_{\mathcal{F}} = \bigcup_{J\in\mathbb{N}^3} \mathscr{A}_J(R)$. Hence,

$$
\begin{aligned}
&\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}}\\ R\subset\Omega}} \sum_{R'\in\mathcal{R}^d_{\mathcal{F}}} |R'|^2 w(R')^{-1} r(R,R') P(R,R') T_{R'}\\
&\quad\le \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}}\\ R\subset\Omega}} \sum_{R'\in\mathscr{A}_{0,0,0}(R)} |R'|^2 w(R')^{-1} r(R,R') P(R,R') T_{R'}\\
&\qquad+ \sum_{j_1\in\mathbb{Z}_+} \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}}\\ R\subset\Omega}} \sum_{R'\in\mathscr{A}_{j_1,0,0}(R)} |R'|^2 w(R')^{-1} r(R,R') P(R,R') T_{R'}\\
&\qquad+ \sum_{j_2\in\mathbb{Z}_+} \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}}\\ R\subset\Omega}} \sum_{R'\in\mathscr{A}_{0,j_2,0}(R)} |R'|^2 w(R')^{-1} r(R,R') P(R,R') T_{R'}\\
&\qquad+ \sum_{j_3\in\mathbb{Z}_+} \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}}\\ R\subset\Omega}} \sum_{R'\in\mathscr{A}_{0,0,j_3}(R)} |R'|^2 w(R')^{-1} r(R,R') P(R,R') T_{R'}\\
&\qquad+ \sum_{j_1,j_2\in\mathbb{Z}_+} \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}}\\ R\subset\Omega}} \sum_{R'\in\mathscr{A}_{j_1,j_2,0}(R)} |R'|^2 w(R')^{-1} r(R,R') P(R,R') T_{R'}\\
&\qquad+ \sum_{j_1,j_2\in\mathbb{Z}_+} \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}}\\ R\subset\Omega}} \sum_{R'\in\mathscr{A}_{j_1,0,j_3}(R)} |R'|^2 w(R')^{-1} r(R,R') P(R,R') T_{R'}\\
&\qquad+ \sum_{j_1,j_3\in\mathbb{Z}_+} \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}}\\ R\subset\Omega}} \sum_{R'\in\mathscr{A}_{j_1,0,j_3}(R)} |R'|^2 w(R')^{-1} r(R,R') P(R,R') T_{R'}\\
&\qquad+ \sum_{j_1,j_2,j_3\in\mathbb{Z}_+} \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}}\\ R\subset\Omega}} \sum_{R'\in\mathscr{A}_{j_1,j_2,j_3}(R)} |R'|^2 w(R')^{-1} r(R,R') P(R,R') T_{R'}\\
&\quad:= I + II + III + IV + V + VI + VII + VIII.
\end{aligned}
$$

In the sequel, we always assume $R, R' \in \mathcal{R}^d_{\mathcal{F}}$ for simplicity. To estimate I, we denote $\mathscr{B}_{0,0,0} = \{R' : 3R' \cap \Omega^{0,0,0} \neq \varnothing\}$. For any $R' \notin \mathscr{B}_{0,0,0}$, we have $3R' \cap \Omega^{0,0,0} = \varnothing$. This implies that $3R' \cap 3R = \varnothing$ for every $R \subset \Omega$ and thus $R' \notin \mathscr{A}_{0,0,0}(R)$. This shows that

$\cup_{R\subset\Omega}\mathscr{A}_{0,0,0}(R)\subset\mathscr{B}_{0,0,0}$. Hence

$$I\le\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{R'\in\mathscr{B}_{0,0,0}}\sum_{\substack{R:R\subset\Omega\\R'\in\mathscr{A}_{0,0,0}(R)}}|R'|^2w(R')^{-1}r(R,R')P(R,R')T_{R'}.$$

For each integer $h\ge1$, let $\mathscr{F}_h^{0,0,0}=\{R'\in\mathscr{B}_{0,0,0},|3R'\cap\Omega^{0,0,0}|\ge 1/2^h|3R'|\}$, $\mathscr{D}_h^{0,0,0}=\mathscr{F}_h^{0,0,0}\backslash\mathscr{F}_{h-1}^{0,0,0}$ and $\Omega_h^{0,0,0}=\bigcup_{R'\in\mathscr{D}_h^{0,0,0}}R'$. Observe that $\mathscr{B}_{0,0,0}=\cup_{h\ge1}\mathscr{D}_h^{0,0,0}$ and that $P(R,R')\le1$ for any pair of rectangles (R,R') with $R'\in\mathscr{B}_{0,0,0}$ and $R'\in\mathscr{A}_{0,0,0}(R)$. Thus

$$I\le\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{h\ge1}\sum_{R'\subset\Omega_h^{0,0,0}}\sum_{\substack{R:R\subset\Omega\\R'\in\mathscr{A}_{0,0,0}(R)}}|R'|^2w(R')^{-1}r(R,R')T_{R'}. \tag{2.9}$$

We now estimate $\sum_{\substack{R:R\subset\Omega,\\R'\in\mathscr{A}_{0,0,0}(R)}}r(R,R')$ for each $h\in\mathbb{Z}_+$ and $R'\subset\Omega_h^{0,0,0}$. Note that $R'\in\mathscr{A}_{0,0,0}(R)$ implies $3R\cap3R'\ne\varnothing$. Using an idea of Chang and R. Fefferman [CF1], for each R, we consider the following eight cases:

Case 1. $|Q_1'|\ge|Q_1|,|Q_2'|\ge|Q_2|,|Q_3'|\ge|Q_3|$;
Case 2. $|Q_1'|\ge|Q_1|,|Q_2'|\ge|Q_2|,|Q_3'|<|Q_3|$;
Case 3. $|Q_1'|\ge|Q_1|,|Q_2'|<|Q_2|,|Q_3'|\ge|Q_3|$;
Case 4. $|Q_1'|<|Q_1|,|Q_2'|\ge|Q_2|,|Q_3'|\ge|Q_3|$;
Case 5. $|Q_1'|\ge|Q_1|,|Q_2'|<|Q_2|,|Q_3'|<|Q_3|$;
Case 6. $|Q_1'|<|Q_1|,|Q_2'|\ge|Q_2|,|Q_3'|<|Q_3|$;
Case 7. $|Q_1'|<|Q_1|,|Q_2'|<|Q_2|,|Q_3'|\ge|Q_3|$;
Case 8. $|Q_1'|<|Q_1|,|Q_2'|<|Q_2|,|Q_3'|<|Q_3|$.

We first consider *Case 1.* In this case, $|R|\le|3R\cap3R'|\le|3R'\cap\Omega^{0,0,0}|\le2^{1-h}|3R'|\le2^{2N+1-h}|R'|$, which implies that $|R'|=2^{h-1-2N+\theta}|R|$ for some integer $\theta\ge0$. For each fixed θ, the number of such R's must be less than $C(\theta+h)^N2^{\theta+h}$. Consequently,

$$\sum_{R\in Case\ 1}r(R,R')\le C\sum_{\theta\ge0}\left(\frac{1}{2^{\theta+h}}\right)^L(\theta+h)^N2^{\theta+h}\le C2^{-hL'},$$

where $L'=L-(N+1)>0$.

We next deal with *Case 2.* We have $|3R'||Q_1\times Q_2|/(2^{2N}|Q_1'\times Q_2'|)\le|3R\cap3R'|\le2^{1-h}|3R'|$, which implies that $|Q_1'\times Q_2'|=2^{h+\theta-1-2N}|Q_1\times Q_2|$ for some integer $\theta\ge0$. For each fixed θ, the number of such $Q_1\times Q_2$'s must be less than $C(\theta+h)^N\cdot2^{\theta+h}$. Similarly, $|Q_3|=2^\lambda|Q_3'|$ for some $\lambda\ge0$. For each λ, $3Q_3\cap3Q_3'\ne\varnothing$ implies that the number of such Q_3's is less than 5^N. It follows that

$$\sum_{R\in Case\ 2}r(R,R')\lesssim\sum_{\theta\ge0}\sum_{\lambda\ge0}\left(\frac{1}{2^{\theta+h+\lambda}}\right)^L(\theta+h)^N2^{\theta+h}\lesssim2^{-hL'}.$$

Cases 3–7 can be handled similarly and the details are left to the reader. We finally handle *Case 8.* We have $|R'|\le|3R\cap3R'|\le|3R'\cap\Omega^{0,0,0}|\le2^{1-h}|3R'|\le2^{1-h+2N}|R'|$, which implies $h \le 2N+1$. Since in this case $|R'| \le |R|$, we have $|R| = 2^\theta |R'|$ for some integer $\theta \ge 0$. For each fixed θ, the number of such R's must be less than 5^N. Therefore,

$$\sum_{R \in Case\ 3} r(R,R') \lesssim \sum_{\theta \ge 0} \left(\frac{1}{2^\theta}\right)^L \lesssim 1.$$

Now we rewrite the right hand side of (2.9) as

$$\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h \ge 1} \sum_{R' \subset \Omega_h^{0,0,0}} \left(\sum_{R \in Case\ 1} + \cdots + \sum_{R \in Case\ 8} \right) r(R,R') \frac{|R'|^2}{w(R')} T_{R'} := I_1 + \cdots + I_8.$$

Note that for $x \in \Omega_h^{0,0,0}$, there exists a dyadic flag rectangle $R \subset \Omega_h^{0,0,0}$ such that $x \in R$. Therefore $\mathcal{M}_{\mathcal{F}}(\chi_{\Omega^{0,0,0}})(x) \ge |3R' \cap \Omega^{0,0,0}|/|3R'| \ge 2^{-h}$. For $q \in (q_w, pL/(2-p))$, we apply the $L^q_w(\mathbb{R}^N)$ boundedness of $\mathcal{M}_{\mathcal{F}}$ and Lemma 2.7 to obtain

$$w(\Omega_h^{0,0,0}) \le w(\{x : \mathcal{M}_{\mathcal{F}}(\chi_{\Omega^{0,0,0}})(x) \ge 2^{-h}\}) \lesssim 2^{qh} w(\Omega^{0,0,0}) \lesssim 2^{qh} w(\Omega).$$

This, together with the estimates in *Cases 1–7*, yields

$$\begin{aligned}
I_1 + \cdots + I_7 &\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h \ge 1} \sum_{R' \subset \Omega_h^{0,0,0}} 2^{-hL'} \frac{|R'|^2}{w(R')} T_{R'} \\
&\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h \ge 1} 2^{-hL'} [w(\Omega_h^{0,0,0})]^{\frac{2}{p}-1} \frac{1}{[w(\Omega_h^{0,0,0})]^{\frac{2}{p}-1}} \sum_{R' \subset \Omega_h^{0,0,0}} \frac{|R'|^2}{w(R')} T_{R'} \\
&\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h \ge 1} 2^{-hL'} (2^{qh})^{\frac{2}{p}-1} [w(\Omega)]^{\frac{2}{p}-1} \sup_{\overline{\Omega}} \frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}} \sum_{R' \subset \overline{\Omega}} \frac{|R'|^2}{w(R')} T_{R'} \\
&\lesssim \sup_{\overline{\Omega}} \frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}} \sum_{R' \subset \overline{\Omega}} \frac{|R'|^2}{w(R')} T_{R'},
\end{aligned}$$

where in the last inequality we have used $\sum_{h \ge 1} 2^{-hL'} (2^{qh})^{\frac{2}{p}-1} \lesssim 1$ for sufficiently large L'.

For I_8, note that in this case, h must be less than $2N+1$. Hence,

$$\begin{aligned}
I_8 &\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{1 \le h \le 2N+1} \sum_{R' \subset \Omega_h^{0,0,0}} \frac{|R'|^2}{w(R')} T_{R'} \\
&\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{1 \le h \le 2N+1} [w(\Omega_h^{0,0,0})]^{\frac{2}{p}-1} \frac{1}{[w(\Omega_h^{0,0,0})]^{\frac{2}{p}-1}} \sum_{R' \subset \Omega_h^{0,0,0}} \frac{|R'|^2}{w(R')} T_{R'} \\
&\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{1 \le h \le 2N+1} (2^{qh})^{\frac{2}{p}-1} [w(\Omega)]^{\frac{2}{p}-1} \sup_{\overline{\Omega}} \frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}} \sum_{R' \subset \overline{\Omega}} \frac{|R'|^2}{w(R')} T_{R'} \\
&\lesssim \sup_{\overline{\Omega}} \frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}} \sum_{R' \subset \overline{\Omega}} \frac{|R'|^2}{w(R')} T_{R'}.
\end{aligned}$$

Combining the estimates above yields

$$I \le C \sup_{\overline{\Omega}} \frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}} \sum_{R' \subset \overline{\Omega}} \frac{|R'|^2}{w(R')} T_{R'}.$$

We now estimate $VIII$. For $J = (j_1, j_2, j_3)$ with $j_1, j_2, j_3 \ge 1$, set

$$a_J = a_{j_1,j_2,j_3} := \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{R \subset \Omega} \sum_{R' \in \mathscr{A}_J(R)} |R'|^2 w(R')^{-1} r(R,R') P(R,R') T_{R'}$$

and $\mathscr{B}_J := \{R' : R'_J \cap \Omega^{0,0,0} \ne \varnothing\}$. For any $R' \notin \mathscr{B}_J$, we have $R'_J \cap \Omega^{0,0,0} = \varnothing$. Thus for every $R \subset \Omega$, it yields $R'_J \cap 3R = \varnothing$, which implies $R' \notin \mathscr{A}_J(R)$ and therefore $\cup_{R \subset \Omega} \mathscr{A}_J(R) \subset \mathscr{B}_J$. Hence,

$$a_J \le \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{R' \in \mathscr{B}_J} \sum_{\substack{R : R \subset \Omega, \\ R' \in \mathscr{A}_J(R)}} |R'|^2 w(R')^{-1} r(R,R') P(R,R') T_{R'}.$$

Let $\mathscr{F}_h^J = \{R' \in \mathscr{B}_J : |R'_J \cap \Omega^{0,0,0}| \ge 1/2^h |R'_J|\}$ for $h \ge 0$, $\mathscr{D}_h^J = \mathscr{F}_h^J \setminus \mathscr{F}_{h-1}^J$ for $h \ge 1$, and $\mathscr{D}_0^J = \varnothing$. Denote $\Omega_h^J = \bigcup_{R' \in \mathscr{D}_h^J} R'$ for $h \ge 1$. Note that $\mathscr{B}_J = \bigcup_{h \ge 1} \mathscr{D}_h^J$. Thus,

$$a_J \le \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h \ge 1} \sum_{R' \in \mathscr{D}_h^J} \sum_{\substack{R : R \subset \Omega, \\ R' \in \mathscr{A}_J(R)}} |R'|^2 w(R')^{-1} r(R,R') P(R,R') T_{R'}. \tag{2.10}$$

Also note that $R' \in \mathscr{A}_J(R)$ implies $|x_{Q_i} - x_{Q'_i}| > [2^{\max_{1 \le k \le i} j_k} \ell(Q'_i)] \vee \ell(Q_i)$ for $i = 1, 2, 3$. Similar to the proof for I, we now consider the following eight cases:

Case 1. $|2^{j_1} Q'_1| \ge |Q_1|, |2^{j_1 \vee j_2} Q'_2| \ge |Q_2|, |2^{j_1 \vee j_2 \vee j_3} Q'_3| \ge |Q_3|$;
Case 2. $|2^{j_1} Q'_1| \ge |Q_1|, |2^{j_1 \vee j_2} Q'_2| \ge |Q_2|, |2^{j_1 \vee j_2 \vee j_3} Q'_3| < |Q_3|$;
Case 3. $|2^{j_1} Q'_1| \ge |Q_1|, |2^{j_1 \vee j_2} Q'_2| < |Q_2|, |2^{j_1 \vee j_2 \vee j_3} Q'_3| \ge |Q_3|$;
Case 4. $|2^{j_1} Q'_1| < |Q_1|, |2^{j_1 \vee j_2} Q'_2| \ge |Q_2|, |2^{j_1 \vee j_2 \vee j_3} Q'_3| \ge |Q_3|$;
Case 5. $|2^{j_1} Q'_1| \ge |Q_1|, |2^{j_1 \vee j_2} Q'_2| < |Q_2|, |2^{j_1 \vee j_2 \vee j_3} Q'_3| < |Q_3|$;
Case 6. $|2^{j_1} Q'_1| < |Q_1|, |2^{j_1 \vee j_2} Q'_2| \ge |Q_2|, |2^{j_1 \vee j_2 \vee j_3} Q'_3| < |Q_3|$;
Case 7. $|2^{j_1} Q'_1| < |Q_1|, |2^{j_1 \vee j_2} Q'_2| < |Q_2|, |2^{j_1 \vee j_2 \vee j_3} Q'_3| \ge |Q_3|$;
Case 8. $|2^{j_1} Q'_1| < |Q_1|, |2^{j_1 \vee j_2} Q'_2| < |Q_2|, |2^{j_1 \vee j_2 \vee j_3} Q'_3| < |Q_3|$.

Rewrite the right hand side of (2.10) as

$$\begin{aligned}&\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h \ge 1} \sum_{R' \in \mathscr{D}_h^J} |R'|^2 w(R')^{-1} T_{R'} \left(\sum_{R \in Case\ 1} + \cdots + \sum_{R \in Case\ 8} \right) r(R,R') P(R,R') \\ &\quad := a_{J,1} + \cdots + a_{J,8}.\end{aligned}$$

We first estimate $a_{J,5}$. For each $h \ge 1$ and $R' \in \mathscr{D}_h^J$, we consider

$$\sum_{\substack{R : R \subset \Omega \\ R' \in \mathscr{A}_J(R)}} r(R,R') P(R,R'). \tag{2.11}$$

Observe that $|Q_1 \times [2^{j_1\vee j_2}Q_2'] \times [2^{j_1\vee j_2\vee j_3}Q_3']| \le |3R_J' \cap 3R|$. Thus

$$\frac{|Q_1|}{|3\cdot 2^{j_1}Q_1'|}|3R_J'| \le |3R_J' \cap 3R| \le |3R_J' \cap \Omega^{0,0,0}| \le \frac{1}{2^{h-1}}|3R_J'|,$$

which yields $2^{h-1}|Q_1| \le 3^{n_1}2^{j_1 n_1}|Q_1'| \le 2^{(j_1+2)n_1}|Q_1'|$. We now consider two subcases.

Subcase 5.1: $|Q_1'| \ge |Q_1|$. In this subcase, since $2^{h-1-j_1 n_1}|Q_1| \lesssim |Q_1'|$, we have $|Q_1'| \approx 2^{h-1-j_1n_1+k}|Q_1|$ for some integer $k \ge 0$. And for each fixed k, the number of such Q_1's must be $\lesssim (k+h)^N 2^{k+h}$.

Subcase 5.2: $|Q_1'| < |Q_1|$. In this subcase, we have $|Q_1'| < |Q_1| \le |2^{j_1}Q_1'|$. So $2^l\ell(Q_1') = \ell(Q_1)$ for some positive integer l satisfying $1 \le l \le j_1$. For each l, the number of Q_1's must be $\lesssim 1$. Moreover, $2^{h-1}2^{ln_1}|Q_1'| = 2^{h-1}|Q_1| \le 2^{(j_1+2)n_1}|Q_1'|$. It follows that $h \le 3n_1 j_1$. Note also that

$$\frac{|x_{Q_1} - x_{Q_1'}|}{\ell(Q_1)} = \frac{|x_{Q_1} - x_{Q_1'}|}{\ell(Q_1')}\frac{\ell(Q_1')}{\ell(Q_1)} \gtrsim 2^{j_1 - l}.$$

In *Case 5*, we also have $|(2^{j_1\vee j_2}Q_2') \times (2^{j_1\vee j_2\vee j_3}Q_3')| \le |Q_2 \times Q_3|$, which implies

$$2^{(j_1\vee j_2)n_2 + (j_1\vee j_2\vee j_3)n_3 + \kappa}|Q_2' \times Q_3'| = |Q_2 \times Q_3|$$

for some $\kappa \ge 0$. And for each fixed κ, the number of such $Q_2 \times Q_3$'s must be $\lesssim 1$. These considerations imply that, for $M > n_1 L$,

$$\begin{aligned}
&\sum_{\textit{Subcase } 5.1} r(R,R')P(R,R') \\
&\quad = \sum_{\textit{Subcase } 5.1} \left(\frac{|Q_1|}{|Q_1'|}\right)^L \left(\frac{|Q_2' \times Q_3'|}{|Q_2 \times Q_3|}\right)^L \left(1 + \frac{|x_{Q_1} - x_{Q_1'}|}{\ell(Q_1')}\right)^{-(n_1+M)} \\
&\qquad\quad \times \left(1 + \frac{|x_{Q_2} - x_{Q_2'}|}{\ell(Q_2)}\right)^{-(n_2+M)} \left(1 + \frac{|x_{Q_3} - x_{Q_3'}|}{\ell(Q_3)}\right)^{-(n_3+M)} \\
&\quad \lesssim \sum_{k,\kappa \ge 0} (k+h)^N 2^{k+h} 2^{-[h+k-j_1n_1]L} 2^{-[(j_1\vee j_2)n_2 + (j_1\vee j_2\vee j_3)n_3 + \kappa]L} 2^{-(n_1+M)j_1} \\
&\quad \lesssim 2^{-h(L-N-1)} 2^{-j_1(M-n_1L)} 2^{-[(j_1\vee j_2)n_2 + (j_1\vee j_2\vee j_3)n_3]L}
\end{aligned}$$

and that, for L,

$$\begin{aligned}
&\sum_{\textit{Subcase } 5.2} r(R,R')P(R,R') = \sum_{\textit{Subcase } 5.2} \left(\frac{|Q_1'|}{|Q_1|}\right)^L \left(\frac{|Q_2' \times Q_3'|}{|Q_2 \times Q_3|}\right)^L \left(1 + \frac{|x_{Q_1} - x_{Q_1'}|}{\ell(Q_1)}\right)^{-(n_1+M)} \\
&\quad \times \left(1 + \frac{|x_{Q_2} - x_{Q_2'}|}{\ell(Q_2)}\right)^{-(n_2+M)} \left(1 + \frac{|x_{Q_3} - x_{Q_3'}|}{\ell(Q_3)}\right)^{-(n_3+M)} \\
&\lesssim \sum_{l=1}^{j_1} \sum_{\kappa \ge 0} 2^{-n_1 l L} 2^{-[(j_1\vee j_2)n_2 + (j_1\vee j_2\vee j_3)n_3 + \kappa]L} 2^{-M(j_1 - l)} \\
&\lesssim 2^{-j_1 L} 2^{-[(j_1\vee j_2)n_2 + (j_1\vee j_2\vee j_3)n_3]L}.
\end{aligned}$$

Now we write

$$a_{J,5} = \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h\geq 1} \sum_{R'\in \mathscr{D}_h^J} |R'|^2 w(R')^{-1} T_{R'} \times \left(\sum_{Subcase\ 5.1} + \sum_{Subcase\ 5.2} \right) r(R,R')P(R,R')$$
$$:= a_{J,5.1} + a_{J,5.2}.$$

Note that if $x \in \Omega_h^J$, then $x \in R$ for some dyadic flag rectangle $R \subset \Omega_h^J$ and therefore $\mathcal{M}_{\mathcal{F}}(\chi_{\Omega^{0,0,0}})(x) \geq |R'_J \cap \Omega^{0,0,0}|/|R'_J| \geq 2^{-h}$. Now we take $L > n_1 q_w(\frac{2}{p}-1) + 2N$, $q \in (q_w, \frac{p(L-2N)}{n_1(2-p)})$ and apply the L_w^q boundedness of $\mathcal{M}_{\mathcal{F}}$ and Lemma 2.7 to get

$$w(\Omega_h^J) \leq w(\{x : \mathcal{M}_{\mathcal{F}}(\chi_{\Omega^{0,0,0}})(x) \geq 2^{-h}\}) \lesssim 2^{qh} w(\Omega^{0,0,0}) \lesssim 2^{qh} w(\Omega).$$

Then

$$\begin{aligned}
a_{J,5.1} &\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h\geq 1} 2^{-h(L-N-1)} 2^{-j_1[M-n_1L]} 2^{-[(j_1\vee j_2)n_2 + (j_1\vee j_2\vee j_3)n_3]L} [w(\Omega_h^J)]^{\frac{2}{p}-1} \\
&\quad \times \frac{1}{[w(\Omega_h^J)]^{\frac{2}{p}-1}} \sum_{R'\subset \Omega_h^J} |R'|^2 w(R')^{-1} T_{R'} \\
&\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h\geq 1} 2^{-h(L-N-1)} 2^{-j_1[M-n_1L]} 2^{-[(j_1\vee j_2)n_2 + (j_1\vee j_2\vee j_3)n_3]L} [2^{qh}]^{\frac{2}{p}-1} [w(\Omega)]^{\frac{2}{p}-1} \\
&\quad \times \sup_{\overline{\Omega}} \frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}} \sum_{R'\subset \overline{\Omega}} |R'|^2 w(R')^{-1} T_{R'} \\
&\lesssim 2^{-j_1[M-n_1L]} 2^{-[(j_1\vee j_2)n_2 + (j_1\vee j_2\vee j_3)n_3]L} \sup_{\overline{\Omega}} \frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}} \sum_{R'\subset \overline{\Omega}} |R'|^2 w(R')^{-1} T_{R'}
\end{aligned}$$

and

$$\begin{aligned}
a_{J,5.2} &\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h=1}^{3n_1j_1} 2^{-j_1L} 2^{-[(j_1\vee j_2)n_2 + (j_1\vee j_2\vee j_3)n_3]L} [w(\Omega_h^J)]^{\frac{2}{p}-1} \\
&\quad \times \frac{1}{[w(\Omega_h^J)]^{\frac{2}{p}-1}} \sum_{R'\subset \Omega_h^J} |R'|^2 w(R')^{-1} T_{R'} \\
&\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} 2^{-j_1(L-(6n_1q/p))} 2^{-[(j_1\vee j_2)n_2 + (j_1\vee j_2\vee j_3)n_3]L} [w(\Omega)]^{\frac{2}{p}-1} \\
&\quad \times \sup_{\overline{\Omega}} \frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}} \sum_{R'\subset \overline{\Omega}} |R'|^2 w(R')^{-1} T_{R'} \\
&\lesssim 2^{-j_1(L-6n_1q/p))} 2^{-[(j_1\vee j_2)n_2 + (j_1\vee j_2\vee j_3)n_3]L} \sup_{\overline{\Omega}} \frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}} \sum_{R'\subset \overline{\Omega}} |R'|^2 w(R')^{-1} T_{R'}.
\end{aligned}$$

Combining these estimates yields that, for $M > n_1 L > 6n_1^2 q/p$,

$$\sum_{j_1,j_2,j_3\geq 1} a_{J,5} \leq \sum_{j_1,j_2,j_3\geq 1} a_{J,5.1} + \sum_{j_1,j_2,j_3\geq 1} a_{J,5.2} \lesssim \sup_{\overline{\Omega}} \frac{1}{|\overline{\Omega}|^{\frac{2}{p}-1}} \sum_{R'\subset \overline{\Omega}} |R'|^2 w(R')^{-1} T_{R'},$$

where we have used the following estimate

$$\sum_{j_1,j_2,j_3\geq 1} 2^{-j_1k_1}2^{-(j_1\vee j_2)k_2}2^{-(j_1\vee j_2\vee j_3)k_3} \leq C \qquad \text{for } k_1,k_2,k_3>0.$$

Using the same skills, we can estimate the other seven terms. Combining these estimates, we obtain

$$VIII = \sum_{j_1,j_2,j_3\geq 1}\Big(a_{J,1}+\cdots+a_{J,8}\Big) \lesssim \sup_{\overline{\Omega}} \frac{1}{|\overline{\Omega}|^{\frac{2}{p}-1}} \sum_{R'\subset\overline{\Omega}} |R'|^2 w(R')^{-1} T_{R'}$$

as desired.

Similarly, the estimates of $II - VII$ can be handled with minor modifications, and hence the proof of Theorem 1.2 follows. $\square$

To show that $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ is the dual spaces of $H^p_{\mathcal{F},w}(\mathbb{R}^N)$, we introduce the multi-parameter flag weighted sequence spaces.

Definition. Let $0<p\leq 1$ and $w\in A^{\mathcal{F}}_\infty(\mathbb{R}^N)$. We use $s^p_w(\mathbb{R}^N)$ to express the collection of all sequences $\{s_R\}$ satisfying

$$\|\{s_R\}\|_{s^p_w(\mathbb{R}^N)} := \Big\|\Big\{\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^J_{\mathcal{F}}} \frac{|s_R|^2}{|R|}\chi_R\Big\}^{\frac12}\Big\|_{L^p_w(\mathbb{R}^N)} < \infty.$$

We also use $c^p_w(\mathbb{R}^N)$ to denote the collection of all sequences $\{t_R\}$ such that

$$\|\{t_R\}\|_{c^p_w(\mathbb{R}^N)} := \sup_{\text{open } \Omega\subset\mathbb{R}^N} \left\{\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{J\in\mathbb{Z}^3}\sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}}\\ R\subset\Omega}} |t_R|^2\frac{|R|}{w(R)}\right\}^{\frac12} < \infty.$$

We will show the duality relationship between s^p_w and c^p_w.

Theorem 2.9. *Let $0<p\leq 1$. Then $(s^p_w(\mathbb{R}^N))^* = c^p_w(\mathbb{R}^N)$. More precisely, for every $\{t_R\}\in c^p_w(\mathbb{R}^N)$, the mapping $\ell_s:\{s_R\}\mapsto\sum_R s_R\bar{t}_R$ defines a continuous linear functional on $s^p_w(\mathbb{R}^N)$ with operator norm $\|\ell_s\|\lesssim\|t\|_{c^p_w(\mathbb{R}^N)}$. Conversely, for every $\ell\in(s^p_w(\mathbb{R}^N))^*$, there is a unique $\{t_R\}\in c^p_w(\mathbb{R}^N)$ such that $\ell(s_R)=\sum_R s_R\bar{t}_R$ and $\|\{t\}_R\|_{c^p_w}\lesssim\|\ell\|$.*

Proof. We first prove $c^p_w(\mathbb{R}^N)\subset(s^p_w(\mathbb{R}^N))^*$. Suppose that $\{t_R\}\in c^p_w(\mathbb{R}^N)$. For $\{s_R\}\in s^p_w(\mathbb{R}^N)$, let $\mathcal{G}(\{s_R\})(x)=\{\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^J_{\mathcal{F}}}|s_R|^2|R|^{-1}\chi_R(x)\}^{\frac12}$. For $i\in\mathbb{Z}$, set $\Omega_i=\{x\in\mathbb{R}^N:\mathcal{G}(\{s_R\})(x)>2^i\}$, $\widetilde{\Omega}_i=\{x\in\mathbb{R}^N:\mathcal{M}_{\mathcal{F}}(\chi_{\Omega_i})(x)>1/2\}$, and $\mathscr{B}_i=\{R\in\mathcal{R}_{\mathcal{F}}:|R\cap\Omega_i|>1/2|R|,\ |R\cap\Omega_{i+1}|\leq 1/2|R|\}$. If $x\in R\in\mathscr{B}_i$, then $\mathcal{M}_{\mathcal{F}}(\chi_{\Omega_i})(x)\geq\frac{1}{|R|}\int_R\chi_{\Omega_i}(y)dy=\frac{|R\cap\Omega_i|}{|R|}>\frac12$, which implies

$$\bigcup_{R\in\mathscr{B}_i} R\subset\widetilde{\Omega}_i. \tag{2.12}$$

Moreover, for $q>q_w$, by the $L^q_w(\mathbb{R}^N)$ boundedness of $\mathcal{M}_{\mathcal{F}}$,

$$w(\widetilde{\Omega}_i)\lesssim w(\Omega_i), \tag{2.13}$$

and by Lemma 2.7,

$$\frac{w(R\cap(\Omega_i\setminus\Omega_{i+1}))}{w(R)}=\frac{w(R\setminus\Omega_{i+1})}{w(R)}\gtrsim\Big(\frac{|R\setminus\Omega_{i+1}|}{|R|}\Big)^q\geq\frac{1}{2^q}. \tag{2.14}$$

Suppose $\{t_R\}\in c^p_w(\mathbb{R}^N)$. By (2.12) $-$ (2.14) and Schwarz's inequality,

$$\begin{aligned}\Big|\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^J_{\mathcal{F}}}s_R\bar{t}_R\Big| &\lesssim \Big|\sum_{i\in\mathbb{Z}}\int_{\widetilde{\Omega}_i\setminus\Omega_{i+1}}\sum_{R\in\mathscr{B}_i}|\bar{t}_R|\frac{|R|^{\frac12}}{w(R)}|s_R||R|^{-\frac12}\chi_R(x)w(x)dx\Big|\\
&\leq\sum_{i\in\mathbb{Z}}\Big\{\sum_{R\subset\widetilde{\Omega}_i}|t_R|^2\frac{|R|}{w(R)}\Big\}^{\frac12}\Big\{\int_{\widetilde{\Omega}_i\setminus\Omega_{i+1}}\sum_{R\in\mathscr{B}_i}\frac{|s_R|^2}{|R|}\chi_R(x)w(x)dx\Big\}^{\frac12}\\
&\lesssim\|\{t_R\}\|_{c^p_w}\sum_{i\in\mathbb{Z}}[w(\widetilde{\Omega}_i)]^{(\frac2p-1)\frac12}\Big\{\int_{\widetilde{\Omega}_i\setminus\Omega_{i+1}}[\mathcal{G}(\{s_R\})(x)]^2w(x)dx\Big\}^{\frac12}\\
&\lesssim\|\{t_R\}\|_{c^p_w}\sum_{i\in\mathbb{Z}}2^i[w(\Omega_i)]^{\frac1p}\\
&\lesssim\|\{t_R\}\|_{c^p_w}\|\mathcal{G}(\{s_R\})\|_{L^p_w}=\|\{t_R\}\|_{c^p_w}\|\{s_R\}\|_{s^p_w},\end{aligned}$$

which implies the inclusion $c^p_w(\mathbb{R}^N)\subset(s^p_w(\mathbb{R}^N))^*$.

For the converse, we assume that $\ell\in(s^p_w(\mathbb{R}^N))^*$. Then it is clear that $\ell(\{s_R\})=\sum_R s_R\bar{t}_R$ for some $\{t_R\}$. Now fix an open set $\Omega\subset\mathbb{R}^N$ with $w(\Omega)<\infty$. Let μ be a measure of $\mathcal{R}_{\mathcal{F}}$ such that $\mu(R)=[w(\Omega)]^{1-2/p}|R|[w(R)]^{-1}$ if $R\subset\Omega$ and otherwise $\mu(R)=0$. Set

$$\|\{s_R\}\|_{\ell^2(\Omega,\mu)}=\Big\{\sum_{J\in\mathbb{Z}^3}\sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}}\\R\subset\Omega}}|s_R|^2[w(\Omega)]^{1-2/p}\frac{|R|}{w(R)}\Big\}^{\frac12}.$$

Then

$$\begin{aligned}&\Big\{\frac{1}{[w(\Omega)]^{\frac2p-1}}\sum_{J\in\mathbb{Z}^3}\sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}}\\R\subset\Omega}}|t_R|^2\frac{|R|}{w(R)}\Big\}^{\frac12}\\
&\quad=\|\{t_R\}\|_{\ell^2(\Omega,\mu)}=\sup_{\|\{s_R\}\|_{\ell^2(\Omega,\mu)}\leq1}\Big|\sum_{J\in\mathbb{Z}^3}\sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}}\\R\subset\Omega}}s_R\bar{t}_R[w(\Omega)]^{1-2/p}\frac{|R|}{w(R)}\Big|\\
&\quad\leq\|\ell\|\sup_{\|\{s_R\}\|_{\ell^2(\Omega,\mu)}\leq1}\Big\|s_R[w(\Omega)]^{1-2/p}\frac{|R|}{w(R)}\Big\|_{s^p_w},\end{aligned}$$

where $\{s_R\}$ satisfies $s_R=0$ if R is not contained in Ω. However, for such $\{s_R\}$, Hölder's inequality yields

$$\begin{aligned}&\Big\|s_R[w(\Omega)]^{1-2/p}\frac{|R|}{w(R)}\Big\|_{s^p_w(\mathbb{R}^N)}\\
&\quad=\Big\{\int_\Omega\Big[\sum_{J\in\mathbb{Z}^3}\sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}}\\R\subset\Omega}}|s_R|^2[w(\Omega)]^{2-4/p}\frac{|R|}{w(R)^2}\chi_R(x)\Big]^{\frac p2}w(x)dx\Big\}^{\frac1p}\end{aligned}$$

$$\leq [w(\Omega)]^{\frac{1}{p}-\frac{1}{2}}\Big\{\int_\Omega \sum_{J\in\mathbb{Z}^3}\sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}}\\ R\subset\Omega}} |s_R|^2[w(\Omega)]^{2-4/p}\frac{|R|}{w(R)^2}\chi_R(x)w(x)dx\Big\}^{\frac{1}{2}}$$
$$= \|\{s_R\}\|_{\ell^2(\Omega,\mu)} \leq 1,$$

which shows $\|\{t_R\}\|_{c^p_w(\mathbb{R}^N)} \leq \|\ell\|$ and thus $\{t_R\}\in c^p_w(\mathbb{R}^N)$. □

Now we define a *lifting operator* $\mathcal{L}$ on $\mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$ and a *projection operator* $\mathcal{T}$ on sequence spaces by

$$\mathcal{L}(f) := \{|R|^{\frac{1}{2}}\psi_J * f(x_R)\} \qquad \text{for } f\in\mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$$

and

$$\mathcal{T}(\{t_R\})(x) := \sum_{J\in\mathbb{Z}^3}\sum_{\mathcal{R}\in\mathcal{R}^J_{\mathcal{F}}} |R|^{\frac{1}{2}}\psi_J(x-x_R)t_R,$$

where $\{\psi_J\}$ satisfies (1.3) and (1.4).

To prove Theorem 1.3, we need the following

Theorem 2.10. *The lifting operator $\mathcal{L}$ is bounded from $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ to $s^p_w(\mathbb{R}^N)$ and bounded from $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ to $c^p_w(\mathbb{R}^N)$. The projection operator $\mathcal{T}$ is bounded from $s^p_w(\mathbb{R}^N)$ to $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ and bounded from $c^p_w(\mathbb{R}^N)$ to $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$. Moreover, $\mathcal{T}\circ\mathcal{L}$ is the identity both on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ and $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$.*

Proof. The boundedness of $\mathcal{L}$ from $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ to $s^p_w(\mathbb{R}^N)$ and from $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ to $c^p_w(\mathbb{R}^N)$ follows directly from Definition of $\mathcal{L}$ and $\mathcal{T}$.

We next show that $\mathcal{T}$ is bounded from $s^p_w(\mathbb{R}^N)$ to $H^p_{\mathcal{F},w}(\mathbb{R}^N)$. By Definition,

$$\|\mathcal{T}(\{t_R\})\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)} = \Big\|\Big\{\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^J_{\mathcal{F}}} |\psi_J * \mathcal{T}(\{t_R\})(x_R)|^2\chi_R\Big\}^{\frac{1}{2}}\Big\|_{L^p_w(\mathbb{R}^N)}.$$

A similar argument to the proof of Theorem 1.1 yields

$$\|\mathcal{T}(\{t_R\})\|_{H^p_{\mathcal{F},w}} \lesssim \Big\|\Big\{\sum_{J'\in\mathbb{Z}^3}\{\mathcal{M}_{\mathcal{F}}[\sum_{R'\in\mathcal{R}^{J'}_{\mathcal{F}}} t^2_{R'}|R'|^{-1}\chi_{R'}]^{\delta/2}\}^{\frac{2}{\delta}}\Big\}^{\frac{1}{2}}\Big\|_{L^p_w}$$
$$\lesssim \Big\|\Big\{\sum_{J'\in\mathbb{Z}^3}\sum_{R'\in\mathcal{R}^{J'}_{\mathcal{F}}} t^2_{R'}|R'|^{-1}\chi_{R'}\Big\}^{\frac{1}{2}}\Big\|_{L^p_w} = \|s\|_{s^p_w}.$$

Finally, we prove that the operator $\mathcal{T}$ is bounded from $c^p_w(\mathbb{R}^N)$ to $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$. Suppose $\{t_R\}\in c^p_w(\mathbb{R}^N)$. Then, for any open set $\Omega\subset\mathbb{R}^N$ with $w(\Omega)<\infty$,

$$\sum_{J\in\mathbb{Z}^3}\sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}}\\ R\subset\Omega}} |t_R|^2\frac{|R|}{w(R)} \leq C[w(\Omega)]^{\frac{2}{p}-1}.$$

Therefore,

$$\begin{aligned}&\sum_{J\in\mathbb{Z}^3}\sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}}\\ R\subset\overline{\Omega}}}|\psi_J*\mathcal{T}(\{t_R\})(x_R)|^2\frac{|R|^2}{w(R)}\\ &\quad=\sum_{J\in\mathbb{Z}^3}\sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}}\\ R\subset\overline{\Omega}}}\Big(\sum_{J'\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^{J'}_{\mathcal{F}}}|\psi_J*\psi_{J'}(x_R-x_{R'})|\cdot t_{R'}\cdot|R'|^{\frac12}\Big)^2\frac{|R|^2}{w(R)}.\end{aligned}$$

Repeating the same argument as in Theorem 1.2 implies

$$\|\mathcal{T}(\{t_R\})\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}\lesssim\sup_{\overline{\Omega}}\Big\{\frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}}\sum_{R'\subset\overline{\Omega}}|t_{R'}|^2\frac{|R'|^2}{w(R')}\Big\}^{\frac12}\approx\|\{t_R\}\|_{c^p_w(\mathbb{R}^N)}.$$

The fact that $\mathcal{T}\circ\mathcal{L}$ is the identity both on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ and $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ follows directly from the discrete Calderón identity in Theorem 2.1. Hence the proof follows. $\square$

Now, we are ready to give

Proof of Theorem 1.3. We first prove the inclusion $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)\subset(H^p_{\mathcal{F},w}(\mathbb{R}^N))^*$. Let $g\in CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$. For $f\in\mathcal{S}_\infty(\mathbb{R}^N)$, define the mapping $\ell_g(f):=\langle f,g\rangle$. By Theorems 2.1, 2.9 and 2.10, we obtain

$$\begin{aligned}|\ell_g(f)|=|\langle f,g\rangle|&=\Big|\Big\langle\sum_{J\in\mathbb{Z}^3}\sum_{R\subset\mathcal{R}^J_{\mathcal{F}}}|R|\psi_J(\cdot-x_R)\psi_J*f(x_R),g\Big\rangle\Big|\\ &=\Big|\sum_{J\in\mathbb{Z}^3}\sum_{R\subset\mathcal{R}^J_{\mathcal{F}}}|R|^{\frac12}\psi_J*f(x_R)|R|^{\frac12}\psi_J*g(x_R)\Big|\\ &=|\langle\mathcal{L}(f),\mathcal{L}(g)\rangle|\lesssim\|\mathcal{L}(f)\|_{s^p_w(\mathbb{R}^N)}\|\mathcal{L}(g)\|_{c^p_w(\mathbb{R}^N)}\\ &\lesssim\|f\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}\|g\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)},\end{aligned}$$

where we have chosen $\psi^{(1)}(-x)=\psi^{(1)}(x)$ and $\psi^{(2)}(-x)=\psi^{(2)}(x)$. Since $\mathcal{S}_\infty(\mathbb{R}^N)$ is dense in $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ (by Corollary 2.6), this implies that the mapping $\ell_g(f)=\langle f,g\rangle$ can be extended to a continuous linear functional on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ and $\|\ell_g\|\le C\|g\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}$.

Conversely, let $\ell\in(H^p_{\mathcal{F},w}(\mathbb{R}^N))^*$ and $\ell_1=\ell\circ\mathcal{T}$. For $\{s_R\}\in s^p_w(\mathbb{R}^N)$, Theorem 2.10 gives

$$|\ell_1(\{s_R\})|=|\ell(\mathcal{T}(\{s_R\}))|\le\|\ell\|\cdot\|\mathcal{T}(\{s_R\})\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}\le C\|\ell\|\cdot\|\{s_R\}\|_{s^p_w(\mathbb{R}^N)},$$

which implies that $\ell_1\in(s^p_w(\mathbb{R}^N))^*$. Then by Theorem 2.9, there exists $\{t_R\}\in c^p_w(\mathbb{R}^N)$ such that $\ell_1(\{s_R\})=\sum_R s_R\bar{t}_R$ for all $\{s_R\}\in s^p_w(\mathbb{R}^N)$ and $\|\{t_R\}\|_{c^p_w(\mathbb{R}^N)}\lesssim\|\ell_1\|\lesssim\|\ell\|$. By Theorem 2.9 again, $\ell=\ell\circ\mathcal{T}\circ\mathcal{L}=\ell_1\circ\mathcal{L}$. Hence

$$\ell(f)=\ell_1(\mathcal{L}(f))=\langle\mathcal{L}(f),t\rangle=\langle f,g\rangle,$$

where $g=\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^J_{\mathcal{F}}}|R|^{\frac12}\ t_R\ \psi_J(x_R-x)$. This implies that $\ell=\ell_g$ and, by Theorem 2.9,

$$\|g\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}\le C\|\{t_R\}\|_{c^p_w(\mathbb{R}^N)}\le C\|\ell_g\|.$$

This finishes the proof of Theorem 1.3. □

3. Weighted boundedness of singular integrals with flag kernels

This section is devoted to proving the boundedness results given in Theorems 1.4, 1.5 and 1.6 for flag singular integrals. To prove Theorem 1.4, we need the following orthogonality estimates.

Lemma 3.1. *Let* $\in \mathcal{S}(\mathbb{R}^N)$ *satisfy*

$$\int_{\mathbb{R}^{n_i}} \varphi(x_1, x_2, x_3)dx_i = 0 \qquad \text{for} \;\; i = 1, 2, 3, \tag{3.1}$$

and define φ_J *by* $\varphi_J(x) := 2^{-j_1n_1+j_2n_2+j_3n_3}\varphi(2^{-j_1}x_1, 2^{-j_2}x_2, 2^{-j_3}x_3)$. *Also let* $\psi_{J'} \in \mathcal{S}(\mathbb{R}^n)$ *be defined in Section 1. Then there exists* $\epsilon > 0$ *such that, for any* $M > 0$,

$$|\varphi_J * \psi_{J'}(x)| \lesssim 2^{-\epsilon(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)}\Big[\prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{j_k'M}}{(\max_{1\le k\le i} 2^{j_k'} + |x_i|)^{n_i+M}}\Big]. \tag{3.2}$$

Proof. There are 8 cases.

Case 1. $j_1 \le j_1', j_2 \le j_2', j_3 \le j_3'$. By (3.1),

$$\begin{aligned}\Big|\varphi_J * \psi^{(1)}_{j_1}(x)\Big| &= \Big|\int_{\mathbb{R}^N} \varphi_J(u)[\psi^{(1)}_{j_1}(x-u) - \psi^{(1)}_{j_1}(x)]dy\Big| \\ &\lesssim 2^{-|j_1-j_1'|}\frac{2^{(j_1\vee j_1')M}}{(2^{j_1\vee j_1'} + |x_1|)^{M+n_1}}\frac{2^{(j_2\vee j_1')M}}{(2^{j_2\vee j_1'} + |x_2|)^{M+n_2}}\frac{2^{(j_3\vee j_1')M}}{(2^{j_3\vee j_1'} + |x_3|)^{M+n_3}}.\end{aligned} \tag{3.3}$$

This together with

$$|\psi^{(2)}_{j_2'} *_2 \psi^{(3)}_{j_3'}(x_2, x_3)| \lesssim \frac{2^{j_2'M}}{(2^{j_2'} + |x_2|)^{M+n_2}}\frac{2^{(j_2'\vee j_3')M}}{(2^{j_2'\vee j_3'} + |x_3|)^{M+n_3}}$$

yields

$$\begin{aligned}|\varphi_J * \psi_{J'}(x)| &= |[\varphi_J * \psi^{(1)}_{j_1}] *_2 [\psi^{(2)}_{j_2'} *_2 \psi^{(3)}_{j_3'}](x)| \\ &\lesssim 2^{-|j_1-j_1'|}\Big[\prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{(j_k\vee j_k')M}}{(\max_{1\le k\le i} 2^{j_k\vee j_k'} + |x_i|)^{n_i+M}}\Big] \\ &= 2^{-|j_1-j_1'|}\Big[\prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{j_k'M}}{(\max_{1\le k\le i} 2^{j_k'} + |x_i|)^{n_i+M}}\Big].\end{aligned} \tag{3.4}$$

The same techniques yield

$$\begin{aligned}|\varphi_J * \psi_{J'}(x)| &= |[\varphi_J *_{2,3} \psi^{(2)}_{j_2}] * [\psi^{(1)}_{j_1'} *_3 \psi^{(3)}_{j_3'}](x)| \\ &\lesssim 2^{-|j_2-j_2'|}\Big[\prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{j_k'M}}{(\max_{1\le k\le i} 2^{j_k'} + |x_i|)^{n_i+M}}\Big]\end{aligned} \tag{3.5}$$

and

$$|\varphi_J * \psi_{J'}(x)| = |[\varphi_J *_3 \psi^{(3)}_{j_3}] * [\psi^{(1)}_{j'_1} *_{2,3} \psi^{(2)}_{j'_2}](x)|$$

(3.6)
$$\lesssim 2^{-|j_3-j'_3|}\Big[\prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{j'_k M}}{(\max_{1\le k\le i} 2^{j'_k} + |x_i|)^{n_i+M}}\Big].$$

Taking the geometric mean of (3.4) − (3.6), we obtain (3.2) with $\epsilon = 1/3$.

Case 2. $j_1 > j'_1, j_2 \le j'_2, j_3 \le j'_3$. We use the moment condition of $\psi^{(1)}$ and Taylor's remainder theorem to get

$$\begin{aligned}
|\varphi_J * \psi^{(1)}_{j_1}(x)| &= \Big|\int_{\mathbb{R}^N} [\varphi_J(x-y) - P_{L-1}[\psi_J](x)]\psi^{(1)}_{j_1}(y)dy\Big| \\
&\lesssim \Big(\sum_{L_1+L_2=L} 2^{-|j_1-j'_1|L_1} 2^{-|j_2-j'_1|L_2}\Big) \\
&\quad \times \frac{2^{(j_1\vee j'_1)M}}{(2^{j_1\vee j'_1}+|x_1|)^{M+n_1}} \frac{2^{(j_2\vee j'_1)M}}{(2^{j_2\vee j'_1}+|x_2|)^{M+n_2}} \frac{2^{(j_3\vee j'_1)M}}{(2^{j_3\vee j'_1}+|x_3|)^{M+n_3}} \\
&\lesssim 2^{-|j_1-j'_1|L} \frac{2^{(j_1\vee j'_1)M}}{(2^{j_1\vee j'_1}+|x_1|)^{M+n_1}} \frac{2^{(j_2\vee j'_1)M}}{(2^{j_2\vee j'_1}+|x_2|)^{M+n_2}} \frac{2^{(j_3\vee j'_1)M}}{(2^{j_3\vee j'_1}+|x_3|)^{M+n_3}},
\end{aligned}$$

where in the last inequality we have used the fact that $|j_1 - j'_1| \ge |j_2 - j'_1|$ and $P_{L-1}[f]$ is the $(L-1)$-th order Taylor's polynomial of f. It follows that

$$\begin{aligned}
|\varphi_J * \psi_{J'}(x)| &= |[\varphi_J * \psi^{(1)}_{j_1}] *_2 [\psi^{(2)}_{j'_2} *_2 \psi^{(3)}_{j'_3}](x)| \\
&\lesssim 2^{-L|j_1-j'_1|} \frac{2^{j_1 M}}{(2^{j_1}+|x_1|)^{M+n_1}} \frac{2^{(j'_1\vee j'_2)M}}{(2^{j'_1\vee j'_2}+|x_2|)^{M+n_2}} \frac{2^{(j'_1\vee j'_2\vee j'_3)M}}{(2^{j'_1\vee j'_2\vee j'_3}+|x_3|)^{M+n_3}} \\
&\le 2^{-(L-M)|j_1-j'_1|}\Big[\prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{j'_k M}}{(\max_{1\le k\le i} 2^{j'_k} + |x_i|)^{n_i+M}}\Big].
\end{aligned}$$

The other cases, $\{j_1 \le j'_1, j_2 > j'_2, j_3 \le j'_3\}$, $\{j_1 > j'_1, j_2 > j'_2, j_3 \le j'_3\}$, $\{j_1 \le j'_1, j_2 \le j'_2, j_3 < j'_3\}$, $\{j_1 > j'_1, j_2 \le j'_2, j_3 < j'_3\}$, $\{j_1 \le j'_1, j_2 > j'_2, j_3 < j'_3\}$, and $\{j_1 > j'_1, j_2 > j'_2, j_3 < j'_3\}$, can be handled by the same manner and details are left to the reader. □

Lemma 3.2. *Let $\mathcal{K}$ be a flag kernel. We have*

(3.7)
$$|\psi_J * \mathcal{K} * \psi_{J'}(x)| \lesssim 2^{-10M(|j_1-j'_1|+|j_2-j'_2|+|j_3-j'_3|)} \prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{j'_k M}}{(\max_{1\le k\le i} 2^{j'_k} + |x_i|)^{1+M}}.$$

Proof. It is well known that $\psi^{(i)}_{j_i} * \psi^{(i)}_{j'_i}$ satisfies the same differential inequalities and moment conditions as $2^{-L|j_i-j'_i|}\psi^{(i)}_{j_i\vee j'_i}$ on $\mathbb{R}^{N_i}$. Thus, $\psi_J * \psi_{J'} = [\psi^{(1)}_{j_1} * \psi^{(1)}_{j'_1}] *_{2,3} [\psi^{(2)}_{j_2} * \psi^{(2)}_{j'_2}] *_3 [\psi^{(3)}_{j_3} * \psi^{(3)}_{j'_3}]$ satisfies the same properties as $2^{-L(|j_1-j'_1|+|j_2-j'_2|+|j_3\vee j'_3|)}\psi_{J\vee J'}$, where

$$\psi_{J\vee J'} := \psi^{(1)}_{j_1\vee j'_1} *_{2,3} \psi^{(2)}_{j_2\vee j'_2} *_3 \psi^{(3)}_{j_3\vee j'_3}.$$

By [NRS, Corollary 2.4.4], $\mathcal{K} = \sum_{j_1\le j_2\le j_3} \varphi^{(J)}_J$, where $\{\varphi^{(J)}\}$ is a bounded collection of C^∞ functions, each of which is supported on $\{|x_i| \le c, i = 1,2,3\}$ with (3.1), and the series converges in the sense of distributions. Lemma (3.1) yields

$$
\begin{aligned}
|\psi_J * \mathcal{K} * \psi_{J'}(x)| &\le \sum_{j_1''\le j_2''\le j_3''} \left|\varphi_{J''}^{(J'')} * [\psi_J * \psi_{J'}](x)\right| \\
&\lesssim \sum_{j_1''\le j_2''\le j_3''} 2^{-L(|j_1''-(j_1\vee j_1')|+|j_2''-(j_2\vee j_2')|+|j_3''-(j_3\vee j_3')|)} \\
&\quad \times 2^{-L(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)} \prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{(j_k\vee j_k')M}}{(\max_{1\le k\le i} 2^{j_k\vee j_k'} + |x_i|)^{1+M}} \\
&\lesssim 2^{-(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)L} \Big[\prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{(j_k\vee j_k')M}}{(\max_{1\le k\le i} 2^{(j_k\vee j_k')} + |x_i|)^{n_i+M}}\Big].
\end{aligned}
$$

This finishes the proof of Lemma 3.7. □

We now turn to the

Proof of Theorem 1.4. By the discrete Calderón reproducing formula,

$$
\begin{aligned}
\|T_{\mathcal{F}}(f)\|_{H^p_{\mathcal{F},w}} &= \Big\|\Big\{\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}_{\mathcal{F}}^J} |\psi_J * \mathcal{K} * f(x_R)|^2 \chi_R\Big\}^{\frac12}\Big\|_{L^p_w} \\
&= \Big\|\Big\{\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}_{\mathcal{F}}^J} \Big|\sum_{J'\in\mathbb{Z}^3}\sum_{R'\in\mathcal{R}_{\mathcal{F}}^J} |R'|\psi_{J'} * f(x_{R'})\psi_J * \mathcal{K} * \psi_{J'}(x_R - x_{R'})\Big|^2 \chi_R\Big\}^{\frac12}\Big\|_{L^p_w}.
\end{aligned}
$$

Lemma 3.7 says that, for each $J, J' \in \mathbb{Z}^3$, $\psi_J * \mathcal{K} * \psi_{J'}$ satisfies the same orthogonality estimate as $\psi_J * \psi_{J'}$. Thus, repeating the same argument as the proof of Theorem 1.1, we obtain

$$
\|Tf\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)} \lesssim \Big\|\Big\{\sum_{J'\in\mathbb{Z}^3}[\mathcal{M}_{\mathcal{F}}(\sum_{R'\in\mathcal{R}_{\mathcal{F}}^J} |\psi_{J'} * f(x_{R'})|^2\chi_{R'})^{\frac{\delta}{2}}]^{\frac{2}{\delta}}\Big\}^{\frac12}\Big\|_{L^p_w(\mathbb{R}^N)} \lesssim \|f\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}.
$$

This concludes the proof of Theorem 1.4. □

To prove Theorem 1.5, we need a new Calderón type identity in terms of bump functions. More precisely, let $\phi^{(i)} \in \mathcal{S}(\mathbb{R}^{N_i})$ be supported on $B(0,2)$ and satisfy

$$
\int_{\mathbb{R}^{N_i}} \phi^{(i)}(x^i)(x^i)^{\alpha_i} dx^i = 0 \qquad \text{for } 0 \le |\alpha| \le M_0 \text{ and } i = 1,2,3,
$$

where M_0 is a large positive integer given in Theorem 3.3 below, and

$$
\sum_{j_i\in\mathbb{Z}} \widehat{\phi^{(i)}}(2^{j_i}\xi^i) = 1 \qquad \text{for } \xi^i \in \mathbb{R}^{N_i}\backslash\{\mathbf{0}\}.
$$

For $J = (j_1, j_2, j_3) \in \mathbb{Z}^3$, set $\phi_J(x) = (\widetilde{\phi_{j_1}^{(1)}} * \widetilde{\phi_{j_2}^{(2)}} * \widetilde{\phi_{j_3}^{(3)}})(x)$, where $\widetilde{\phi_{j_i}^{(i)}} = \delta_{\mathbb{R}^{N-N_i}} \otimes \phi_{j_i}^{(i)}$.

Theorem 3.3. *Let $0 < p \le 1$ and $w \in A_\infty^{\mathcal{F}}(\mathbb{R}^N)$. Suppose $M_0 \ge 10(N\{[q_w/p-1]\vee 2\}+1)$ (here $[\cdot]$ means the greatest integer function). For a fixed sufficiently large $K \in \mathbb{N}$, let*

$\mathcal{R}_{\mathcal{F}}^{J,K} = \mathcal{R}_{\mathcal{F}}^{j_1-K,j_2-K,j_3-K}$ *and let* x_R *denote the left-lower corner of* R. *Then, for every* $f \in L^2(\mathbb{R}^N) \cap H^p_{\mathcal{F},w}$ *there exists* $h \in L^2(\mathbb{R}^N) \cap H^p_{\mathcal{F},w}$ *depending only on* f *such that*

$$f(x) \stackrel{L^2}{=} \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} |R|\phi_J(x-x_R)\phi_J * h(x_R). \tag{3.8}$$

Moreover,

$$\|f\|_{H^p_{\mathcal{F},w}} \approx \|h\|_{H^p_{\mathcal{F},w}}. \tag{3.9}$$

Proof. For $f \in L^2(\mathbb{R}^N)$, applying the Fourier transform gives $f = \sum_{J\in\mathbb{Z}^3} \phi_J * \phi_J * f$, where the series converges in $L^2(\mathbb{R}^N)$ norm. Using Coifman's idea of the decomposition of the identity operator, we have

$$\begin{aligned} f(x) = &\sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} |R|\phi_J * f(x_R)\phi_J(x-x_R) \\ &+ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} \int_R [\phi_J(x-x')(\phi_J * f)(x') - \phi_J(x-x_R)(\phi_J * f)(x_R)]dx' \\ := &\, T_K(f)(x) + R_K(f)(x), \end{aligned}$$

where K is a fixed large integer to be determined later.

We can decompose $R_K(f)$ further as

$$\begin{aligned} R_K(f)(x) = &\sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} \int_R [\phi_J(x-x') - \phi_J(x-x_R)](\phi_J * f)(x')dx' \\ &+ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} \int_R \phi_J(x-x')[(\phi_J * f)(x') - (\phi_J * f)(x_R)]dx' \\ := &\, R_K^1(f)(x) + R_K^2(f)(x). \end{aligned}$$

We claim that for $k = 1, 2$,

$$\|R_K^k(f)\|_{H^p_{\mathcal{F},w}} \le C2^{-K}\|f\|_{H^p_{\mathcal{F},w}}, \tag{3.10}$$

where C is a constant independent of f, K and x_R.

Assume the claim for the moment. Then choosing sufficiently large K such that $C2^{-K} < 1$ implies that both T_K and $T_K^{-1} = \sum_{n=0}^{\infty}(R_K)^n$ are bounded on $L^2(\mathbb{R}^N)$ and on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$. Setting $h = R_K^{-1}(f)$ gives (3.9). Moreover,

$$f = T_K(T_K^{-1}(f)) = \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} |R|\phi_J(\cdot - x_R)(\phi_J * h)(x_R),$$

where the series converges in $L^2(\mathbb{R}^N)$.

To finish the proof of Theorem 1.5, it suffices to verify the claim. Since the proofs for R_K^1 and R_K^2 are similar, we only estimate R_K^1. Let $f \in L^2(\mathbb{R}^N) \cap H^p_{\mathcal{F},w}(\mathbb{R}^N)$. The discrete

Calderón reproducing formula in Theorem 2.1 yields

$$\psi_{J'} * R_K^1(f)(x) = \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} \int_R \psi_{J'} * [\phi_J(\cdot - x') - \phi_J(\cdot - x_R)](x)(\phi_J * f)(x')dx'$$

$$(3.11) \qquad = \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} \int_R \psi_{J'} * [\phi_J(\cdot - x') - \phi_J(\cdot - x_R)](x) \times \Big(\sum_{J''\in\mathbb{Z}^3} \sum_{R''\in\mathcal{R}_{\mathcal{F}}^{J''}} |R''| \cdot \psi_{J''} * f(x_{R''})\phi_J * \psi_{J''}(x' - x_{R''})\Big)dx',$$

where $x_{R''} = (x_{Q_1''}, x_{Q_2''}, x_{Q_3''})$ is the left-lower corner of R''. Set $\widetilde{\phi}_J(u) = \phi_J(u - x') - \phi_J(u - x_R)$. Applying Lemma 2.2 with M sufficiently large (which will be determined later) and $L = 10M$, we obtain that for some constant C (depending only on M, ψ and ϕ, but independent of K),

$$|\psi_{J'} * \widetilde{\phi}_J(x)| \le C2^{-K}2^{-10M(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)} \prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{j_k'M}}{(\max_{1\le k\le i} 2^{j_k'} + |x_i - x_i'|)^{1+M}},$$

and, similarly,

$$|\phi_J * \psi_{J''}(x' - x_{R''})| \le C2^{-10M(|j_1-j_1''|+|j_2-j_2''|+|j_3-j_3''|)} \prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{j_k''M}}{(\max_{1\le k\le i} 2^{j_k''} + |x_i' - x_{Q_i''}|)^{1+M}}.$$

Substituting both estimates into the last term of (3.11) yields

(3.12)

$$\begin{aligned} |\psi_{J'} * R_K^1(f)(x)| &\lesssim \sum_{J''\in\mathbb{Z}^3} \sum_{R''\in\mathcal{R}_{\mathcal{F}}^{J''}} |R''||\psi_{J''} * f(x_{R''})| \\ &\quad \times \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} \int_R 2^{-K} \prod_{i=1}^{3} 2^{-|j_i-j_i'|3M} \frac{\max_{1\le k\le i} 2^{j_k'M}}{(\max_{1\le k\le i} 2^{j_k'} + |x_i - x_i'|)^{1+M}} \\ &\quad \times 2^{-|j_i-j_i''|3M} \frac{\max_{1\le k\le i} 2^{j_k''M}}{(\max_{1\le k\le i} 2^{j_k''} + |x_i' - x_{Q_i''}|)^{1+M}} dx' \\ &\lesssim 2^{-K} \sum_{J''\in\mathbb{Z}^3} \sum_{R''\in\mathcal{R}_{\mathcal{F}}^{J''}} 2^{-(|j_1'-j_1''|+|j_2'-j_2''|+|j_3'-j_3''|)M}|R''| \\ &\quad \times \Big(\prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{(j_k'\vee j_k'')M}}{(\max_{1\le k\le i} 2^{j_k'\vee j_k''} + |x_i - x_{Q_i''}|)^{1+M}}\Big)|\psi_{J''} * f(x_{R''})|. \end{aligned}$$

Now we choose $M = N\{[q_w/p - 1] \vee 2\} + 1$, $L = 10M$ and $N/(N + M) < \delta < 1$. Then $p/\delta > q_w$ so that $w \in A_{p/\delta}^{\mathcal{F}}(\mathbb{R}^N)$. Arguing as in the proof of Theorem 1.1 yields

$$\|R_K^1(f)\|_{H_{\mathcal{F},w}^p} \lesssim 2^{-K}\Big\|\Big\{\sum_{J''\in\mathbb{Z}^3} \{\mathcal{M}_{\mathcal{F}}(\sum_{R''\in\mathcal{R}_{\mathcal{F}}^{J''}} |\psi_{J''} * f(x_{R''})|\chi_{R''})^\delta\}^{\frac{2}{\delta}}\Big\}^{\frac{1}{2}}\Big\|_{L_w^p} \lesssim 2^{-K}\|f\|_{H_{\mathcal{F},w}^p}.$$

This confirms claim (3.10) and hence Theorem 3.3 follows. $\square$

Using a similar argument to the one in the proof of Theorem 1.1, one can prove

Corollary 3.4. *Suppose $w \in A_\infty^{\mathcal{F}}(\mathbb{R}^N)$. Then, for $f \in H^p_{\mathcal{F},w}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ and $0 < p \le 1$, we have*

$$\|f\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)} \approx \|\widetilde{g}_{\mathcal{F}}(f)\|_{L^p_w(\mathbb{R}^N)} \equiv \Big\| \Big\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}^{J,K}_{\mathcal{F}}} |\phi_J * h(x_R)|^2 \chi_R \Big\}^{1/2} \Big\|_{L^p_w(\mathbb{R}^N)},$$

where h and K are the same as in Theorem 3.3.

The key to the proof of Theorem 1.5 is the following

Lemma 3.5. *Suppose $0 < p \le 1$ and $w \in A_\infty^{\mathcal{F}}(\mathbb{R}^N)$. If $f \in H^p_{\mathcal{F},w}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, then $f \in L^p_w(\mathbb{R}^N)$ and there is a constant $C_p > 0$ independent of the $L^2(\mathbb{R}^N)$ norm of f such that*

$$\|f\|_{L^p_w(\mathbb{R}^N)} \le C_p \|f\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}.$$

Proof. Without loss of generality, we may assume $w \in A_q^{\mathcal{F}}(\mathbb{R}^N)$ for some $q \in [2,\infty)$. Given $f \in L^2(\mathbb{R}^N) \cap H^p_{\mathcal{F},w}(\mathbb{R}^N)$, set $\Omega_i = \{x \in \mathbb{R}^N : \widetilde{g}_{\mathcal{F}}(h)(x) > 2^i\}$ where h is given by Theorem 4.3, and

$$B_i = \{(J,R) : J \in \mathbb{Z}^3, R \in \mathcal{R}^{J,K}_{\mathcal{F}}, |R\cap\Omega_i| > (1/2)|R|, |R \cap \Omega_{i+1}| \le (1/2)|R|\}.$$

Applying the discrete Calderón reproducing formula in Theorem 3.3, we can write

$$f = \sum_{i\in\mathbb{Z}} \sum_{(J,R)\in B_i} |R| \widetilde{\phi}_J(\cdot - x_R)\phi_J * h(x_R) \qquad \text{for } f \in L^2(\mathbb{R}^N) \cap H^p_{\mathcal{F},w}(\mathbb{R}^N).$$

We claim that

$$\Big\| \sum_{(J,R)\in B_i} |R|\widetilde{\phi}_J(\cdot - x_R)\phi_J * h(x_R) \Big\|^p_{L^p_w(\mathbb{R}^N)} \lesssim 2^{pi} w(\Omega_i). \tag{3.13}$$

Since $0 < p \le 1$, the above claim together with Theorem 4.3 yields

$$\begin{aligned} \|f\|^p_{L^p_w(\mathbb{R}^N)} &\le \sum_{i\in\mathbb{Z}} \Big\| \sum_{(J,R)\in B_i} |R|\widetilde{\phi}_J(\cdot - x_R)\phi_J * h(x_R) \Big\|^p_{L^p_w(\mathbb{R}^N)} \\ &\lesssim \sum_{i\in\mathbb{Z}} 2^{pi} w(\Omega_i) \lesssim \|\widetilde{g}(h)\|^p_{L^p_w(\mathbb{R}^N)} \approx \|h\|^p_{H^p_{\mathcal{F},w}(\mathbb{R}^N)} \approx \|f\|^p_{H^p_{\mathcal{F},w}(\mathbb{R}^N)} \end{aligned}$$

and Lemma 3.5 would follow.

To show claim (3.13), we note that if $(J,R) \in B_i$, then $\widetilde{\phi}_J(x - x_R)$ is supported in $\widetilde{\Omega}_i := \{x : \mathcal{M}_{\mathcal{F}}(\chi_{\Omega_i})(x) > 1/100\}$. By Hölder's inequality,

$$\begin{aligned} &\Big\| \sum_{(J,R)\in B_i} |R|\widetilde{\phi}_J(\cdot - x_R)\phi_J * h(x_R) \Big\|^p_{L^p_w(\mathbb{R}^N)} \\ &\quad\lesssim w(\widetilde{\Omega}_i)^{1-(p/q)} \Big\| \sum_{(J,R)\in B_i} |R|\widetilde{\phi}_J(\cdot - x_R)\phi_J * h(x_R) \Big\|^p_{L^q_w(\mathbb{R}^N)}. \end{aligned} \tag{3.14}$$

We now estimate the last L^q_w-norm by duality argument. For $\zeta \in L^{q'}_{w^{1-q'}}(\mathbb{R}^N)$ with $\|\zeta\|_{L^{q'}_{w^{1-q'}}(\mathbb{R}^N)} \le 1$,

$$
\begin{aligned}
&\Big|\Big\langle \sum_{(J,R)\in B_i} |R|\widetilde{\phi}_J(\cdot - x_R)\phi_J * h(x_R), \zeta\Big\rangle\Big| \\
&\quad = \Big| \sum_{(J,R)\in B_i} \int \overline{\phi}_J * \zeta(x_R)\phi_J * h(x_R)\chi_R(x)dx\Big| \\
&\quad \le \Big\|\Big\{ \sum_{(J,R)\in B_i} |\phi_J * h(x_R)|^2\chi_R\Big\}^{\frac{1}{2}}\Big\|_{L^q_w(\mathbb{R}^N)} \Big\|\Big\{ \sum_{(J,R)\in B_i} |\overline{\phi}_J * \zeta(x_R)|^2\chi_R\Big\}^{\frac{1}{2}}\Big\|_{L^{q'}_{w^{1-q'}}(\mathbb{R}^N)} \\
&\quad := I_1 \times I_2,
\end{aligned}
$$

where $\overline{\phi}_J(x) = \widetilde{\phi}_J(-x)$.

We first estimate I_2. Since $w \in A^{\mathcal{F}}_q(\mathbb{R}^N)$ implies $w^{1-q'} \in A^{\mathcal{F}}_{q'}(\mathbb{R}^N)$, Corollary 3.4 and Remark 1.2 yield

$$
I_2 \lesssim \Big\|\Big\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}^{J,K}_{\mathcal{F}}} |\overline{\phi}_J * \zeta(x_R)|^2\chi_R\Big\}^{\frac{1}{2}}\Big\|_{L^{q'}_{w^{1-q'}}(\mathbb{R}^N)} \approx \|\zeta\|_{L^{q'}_{w^{1-q'}}(\mathbb{R}^N)} \le 1. \tag{3.15}
$$

As for I_1, note that $\Omega_i \subset \widetilde{\Omega}_i$ and $w(\widetilde{\Omega}_i) \lesssim w(\Omega_i)$ due to the $L^q_w(\mathbb{R}^N)$ boundedness of $\mathcal{M}_{\mathcal{F}}$. For any $(J,R) \in B_i$ and $x \in R$, $\mathcal{M}_{\mathcal{F}}(\chi_{R\cap\widetilde{\Omega}_i\setminus\Omega_{i+1}})(x) > \frac{1}{2}$. Applying Corollary 2.5 again, we have

$$
\begin{aligned}
I_1^q &= \int_{\mathbb{R}^N} \Big\{ \sum_{(J,R)\in B_i} |\phi_J * h(x_R)|^2\chi_R(x)\Big\}^{q/2} w(x)\,dx \\
&\lesssim \int_{\mathbb{R}^N} \Big\{ \sum_{(J,R)\in B_i} |\phi_J * h(x_R)\mathcal{M}_{\mathcal{F}}(\chi_{R\cap\widetilde{\Omega}_i\setminus\Omega_{i+1}})(x)|^2\Big\}^{q/2} w(x)\,dx \\
&\lesssim \int_{\widetilde{\Omega}_i\setminus\Omega_{i+1}} \Big\{ \sum_{(J,R)\in B_i} |\phi_J * h(x_R)|^2\chi_R(x)\Big\}^{q/2} w(x)\,dx \\
&\lesssim 2^{iq}w(\widetilde{\Omega}_i) \lesssim 2^{iq}w(\Omega_i).
\end{aligned} \tag{3.16}
$$

Combining both estimates (3.15) and (3.16), we obtain

$$
\Big\| \sum_{(J,R)\in B_i} |R|\widetilde{\phi}_J(\cdot - x_R)\phi_J * h(x_R)\Big\|_{L^q_w(\mathbb{R}^N)} \lesssim 2^{iq}w(\Omega_i).
$$

Plugging this estimate into (3.14) yields claim (3.13), and hence Lemma 3.5 follows. □

Proof of Theorem 1.5. For $f \in L^2(\mathbb{R}^N) \cap H^p_{\mathcal{F},w}(\mathbb{R}^N)$, by Lemma 3.5 and Theorem 1.4,

$$
\|T(f)\|_{L^p_w(\mathbb{R}^N)} \le C\|T(f)\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)} \le C\|f\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}.
$$

Corollary 2.6 together with a limiting argument yields Theorem 1.5. □

Finally, we prove Theorem 1.6. It is known that $L^2(\mathbb{R}^N) \cap H^p_{\mathcal{F},w}(\mathbb{R}^N)$ is dense in $H^p_{\mathcal{F},w}(\mathbb{R}^N)$. This allows us to use the discrete Calderón reproducing formula in Theorem

3.3, which plays a crucial role in the proof of the boundedness of flag singular integral operators on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$. However, $L^2(\mathbb{R}^N)\cap CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ is not dense in $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$. As a substitution, we prove the following lemma, which is called the *weak density.*

Lemma 3.6. *Let* $0<p\le 1$ *and* $w\in A^{\mathcal{F}}_\infty(\mathbb{R}^N)$*. Then* $L^2(\mathbb{R}^N)\cap CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ *is dense in* $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ *in the weak topology* $\langle H^p_{\mathcal{F},w}(\mathbb{R}^N), CMO^p_{\mathcal{F},w}(\mathbb{R}^N)\rangle$*. More precisely, for any* $f\in CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$*, there exists a sequence* $\{f_n\}\subset L^2(\mathbb{R}^N)\cap CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ *satisfying* $\|f_n\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}\lesssim \|f\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}$ *and*

$$\lim_{n\to\infty}\langle f_n,g\rangle=\langle f,g\rangle \qquad \textit{for any } g\in H^p_{\mathcal{F},w}(\mathbb{R}^N).$$

Proof. Suppose $f\in CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$. Set

$$f_n(x)=\sum_{|j|\le n,|k|\le n}\ \sum_{R\subset B(0,n)}|R|\psi_J*f(x_R)\psi_J(x-x_R),$$

where $\{\psi_J\}$ satisfy (1.3) and (1.4). It is easy to see that $f_n\in L^2(\mathbb{R}^N)$. Repeating the same proof as the one in Theorem 1.2, we have $\|f_n\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}\le C\|f\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}$ and hence $f_n\in L^2(\mathbb{R}^N)\cap CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$. For any $g\in\mathcal{S}_\infty(\mathbb{R}^N)$, the discrete Calderón reproducing formula given in Theorem 2.1 yields

$$\begin{aligned}\langle f-f_n,g\rangle&=\Big\langle\sum_{|j|>n, or |k|>n, or R\subsetneq B(0,n)}|R|\psi_J*f(x_R)\psi_J(\cdot-x_R),g\Big\rangle\\&=\Big\langle f,\sum_{|j|>n, or |k|>n, or R\subsetneq B(0,n)}|R|\psi_J*g(x_R)\psi_J(\cdot-x_R)\Big\rangle.\end{aligned}$$

By Corollary 2.6, the function

$$\sum_{|j|>n, or |k|>n, or R\subsetneq B(0,n)}|R|\psi_J*g(x_R)\psi_J(x-x_R)$$

belongs to $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ and its $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ norm tends to 0 as $n\to\infty$. Hence, Theorem 1.3 implies that $\langle f-f_n,g\rangle$ tends to zero as $n\to\infty$. Since $\mathcal{S}_\infty(\mathbb{R}^N)$ is dense in $H^p_{\mathcal{F},w}(\mathbb{R}^N)$, a standard limiting argument finishes the proof of Lemma 3.6. $\square$

Now let us show how a flag singular integral operator $T_{\mathcal{F}}$ acts on $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$. Given $f\in CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$, by Lemma 3.6, there is a sequence $\{f_n\}\subset L^2(\mathbb{R}^N)\cap CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ such that

$$\left\{\begin{array}{ll}\|f_n\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}\le C\|f\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}\\ \lim\limits_{n\to\infty}\langle f_n,g\rangle=\langle f,g\rangle & \text{for any } g\in L^2(\mathbb{R}^N)\cap H^p_{\mathcal{F},w}(\mathbb{R}^N)\end{array}\right. . \tag{3.17}$$

We thus define

$$\langle T_{\mathcal{F}}(f),g\rangle=\lim_{n\to\infty}\langle T_{\mathcal{F}}(f_n),g\rangle \qquad \text{for any } g\in L^2(\mathbb{R}^N)\cap H^p_{\mathcal{F},w}(\mathbb{R}^N).$$

To see that the limit exists, write $\langle (T_{\mathcal{F}})(f_j-f_k),g\rangle=\langle f_j-f_k,(T_{\mathcal{F}})^*(g)\rangle$ since both f_j-f_k and g belong to $L^2(\mathbb{R}^N)$, and $T_{\mathcal{F}}$ is bounded on $L^2(\mathbb{R}^N)$. By Theorem 1.4, $(T_{\mathcal{F}})^*$ is bounded on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$, and thus $(T_{\mathcal{F}})^*(g)\in L^2(\mathbb{R}^N)\cap H^p_{\mathcal{F},w}(\mathbb{R}^N)$. Therefore, by Lemma

3.6, $\langle f_j - f_k, (T_{\mathcal{F}})^*(g)\rangle$ tends to zero as $j, k \to \infty$. It is also easy to verify that the definition of $T_{\mathcal{F}}(f)$ is independent of the choice of the sequence f_n satisfying the conditions in Lemma 3.6.

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. We first show that for $f \in L^2(\mathbb{R}^N) \cap CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ and any open set Ω,

$$\left\{\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{J\in\mathbb{Z}^3} \sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}}\\ R\subset\Omega}} |\psi_J * K_{\mathcal{F}} * f(x_R)|^2 \frac{|R|^2}{w(R)}\right\}^{\frac{1}{2}} \lesssim \|f\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}. \tag{3.18}$$

Using the discrete Calderón reproducing formula given in Theorem 2.1, we write

$$\begin{aligned}
&\sum_{J\in\mathbb{Z}^3} \sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}}\\ R\subset\Omega}} |\psi_J * K_{\mathcal{F}} * f(x_R)|^2 \frac{|R|^2}{w(R)}\\
&\qquad = \sum_{J\in\mathbb{Z}^3} \sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}}\\ R\subset\Omega}} \Big| \sum_{J'\in\mathbb{Z}^3} \sum_{R'\in\mathcal{R}^{J'}_{\mathcal{F}}} t_{R'} |R'|^{1/2} \psi_J * \mathcal{K} * \psi_{J'}(x_R - x_{R'})\Big|^2 \frac{|R|^2}{w(R)},
\end{aligned}$$

where $t_{R'} = \psi_{J'} * f(x_{R'})|R'|^{1/2}$. By (3.7), $\psi_J * \mathcal{K} * \psi_{J'}$ satisfies the same almost orthogonality estimate as $\psi_J * \psi_{J'}$. Repeating the same argument as in Theorem 1.2 yields (3.18).

For $f \in CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$, there is a sequence $\{f_n\} \subset L^2(\mathbb{R}^N) \cap CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ satisfying (3.17). By the definition of $T_{\mathcal{F}}(f)$ and the boundedness of $T_{\mathcal{F}}$ on $L^2(\mathbb{R}^N) \cap CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$,

$$\begin{aligned}
\|T_{\mathcal{F}}(f)\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)} &\le \liminf_{n\to\infty} \|T_{\mathcal{F}}(f_n)\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}\\
&\lesssim \liminf_{n\to\infty} \|f_n\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)} \lesssim \|f\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)},
\end{aligned}$$

which concludes the proof of Theorem 1.6. □

4. Calderón-Zygmund decomposition and interpolation

We first prove the Calderón-Zygmund decomposition for $H^p_{\mathcal{F},w}$.

Proof of Theorem 1.7. According to Corollary 2.6, $L^2(\mathbb{R}^N) \cap H^p_{\mathcal{F},w}(\mathbb{R}^N)$ is dense in $H^p_{\mathcal{F},w}(\mathbb{R}^N)$. Thus it suffices to prove Theorem 1.7 for $f \in L^2(\mathbb{R}^N) \cap H^p_{\mathcal{F},w}(\mathbb{R}^N)$. Given any fixed $\alpha > 0$, let $\Omega_l = \{x \in \mathbb{R}^N : \widetilde{g}_{\mathcal{F}}(f)(x) > \alpha 2^l\}$, $l \in \mathbb{Z}$, where $\widetilde{g}_{\mathcal{F}}(f)(x) := \{\sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}^{J,K}_{\mathcal{F}}} |\phi_J * f(x_R)|^2 \chi_R(x)\}^{1/2}$ and K is given in Theorem 3.3. For $J \in \mathbb{Z}^3$, let $\mathcal{R}^{J,K}_0 = \{R \in \mathcal{R}^{J,K}_{\mathcal{F}} : |R \cap \Omega_0| < 1/2|R|\}$ and

$$\mathcal{R}^{J,K}_l = \{R \in \mathcal{R}^{J,K}_{\mathcal{F}} : |R \cap \Omega_{l-1}| \ge 1/2|R|, |R \cap \Omega_l| < 1/2|R|\} \qquad \text{for } l \ge 1.$$

It follows from Theorem 3.3 that there exists $h \in L^2 \cap H^p_{\mathcal{F},w}$ such that

$$f(x) = \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}^{J,K}_0} |R|\phi_J * h(x_R)\phi_J(x - x_R)$$

$$+\sum_{J\in\mathbb{Z}^3}\sum_{l\geq 1}\sum_{R\in\mathcal{R}_l^{J,K}}|R|\phi_J*h(x_R)\phi_J(x-x_R):=g(x)+b(x).$$

We first estimate $\|g\|_{H^{p_2}_{\mathcal{F},w}}$. Repeating the same argument as in the proof of Theorem 1.1, we deduce that for $(n-1)/(n-1+M)<\delta<\min\{(p_2/q_w),1\}$,

$$\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^J_{\mathcal{F}}}|\psi_J*g(x_R)|^2\chi_R(x)\lesssim\sum_{J'\in\mathbb{Z}^3}\Big\{\mathcal{M}_{\mathcal{F}}\Big[\Big(\sum_{R'\in\mathcal{R}_0^{J',K}}|\phi_{J'}*g(x_{R'})|^2\chi_{R'}\Big)^{\frac{\delta}{2}}\Big](x)\Big\}^{\frac{2}{\delta}}.$$

Take the square root on both sides and apply Corollary 2.5 on $L^{p_2/\delta}_w(\ell^{2/\delta})$ (note that $w\in A^{\mathcal{F}}_{p_2/\delta}$) to derive

$$\|g\|_{H^{p_2}_{\mathcal{F},w}(\mathbb{R}^N)}\lesssim\Big\|\Big\{\sum_{J'\in\mathbb{Z}^3}\sum_{R'\in\mathcal{R}_0^{J',K}}|\phi_{J'}*h(x_{R'})|^2\chi_{R'}\Big\}^{1/2}\Big\|_{L^{p_2}_w(\mathbb{R}^N)}.$$

We claim

$$\int_{\{\widetilde{g}_{\mathcal{F}}(f)(x)\leq\alpha\}}[\widetilde{g}_{\mathcal{F}}(f)(x)]^{p_2}w(x)dx\gtrsim\Big\|\Big\{\sum_{J'\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}_0^{J',K}}|\phi_{J'}*h(x_{R'})|^2\chi_{R'}\Big\}^{\frac{1}{2}}\Big\|^{p_2}_{L^{p_2}_w(\mathbb{R}^N)},\tag{4.1}$$

which implies

$$\begin{aligned}\|g\|^{p_2}_{H^{p_2}_{\mathcal{F},w}(\mathbb{R}^N)}&\lesssim\int_{\{\widetilde{g}_{\mathcal{F}}(f)(x)\leq\alpha\}}[\widetilde{g}_{\mathcal{F}}(f)(x)]^{p_2}w(x)dx\\&\leq\alpha^{p_2-p}\int_{\{\widetilde{g}_{\mathcal{F}}(f)(x)\leq\alpha\}}[\widetilde{g}_{\mathcal{F}}(f)(x)]^{p}w(x)dx\lesssim\alpha^{p_2-p}\|f\|^p_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}\end{aligned}$$

as desired. It suffices to verify claim (4.1). Choose $\delta<\min\{p_2/q_w,1\}$ and get

$$\begin{aligned}&\int_{\{\widetilde{g}_{\mathcal{F}}(f)(x)\leq\alpha\}}[\widetilde{g}_{\mathcal{F}}(f)(x)]^{p_2}w(x)dx\\&\quad=\int_{\mathbb{R}^N}\Big\{\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^{J,K}_{\mathcal{F}}}|\phi_J*h(x_R)|^2\chi_{R\cap\complement\Omega_0}(x)\Big\}^{\frac{p_2}{2}}w(x)dx\\&\quad\gtrsim\int_{\mathbb{R}^N}\Big\{\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^{J,K}_{\mathcal{F}}}(|\phi_J*h(x_R)|^\delta\mathcal{M}_{\mathcal{F}}(\chi_{R\cap\complement\Omega_0})(x))^{\frac{2}{\delta}}\Big\}^{\frac{p_2}{2}}w(x)dx\\&\quad\gtrsim\Big\|\Big\{\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^{J,K}_{\mathcal{F}}}|\phi_J*h(x_R)|^2\chi_R\Big\}^{\frac{1}{2}}\Big\|^{p_2}_{L^{p_2}_w(\mathbb{R}^N)},\end{aligned}$$

where in the last inequality we have used the estimate that $\chi_R(x)\leq 2^{\frac{1}{\delta}}\mathcal{M}_{\mathcal{F}}(\chi_{R\cap\complement\Omega_0})^{\frac{1}{\delta}}(x)$ for $R\in\mathcal{R}^{J,K}_{\mathcal{F}}$, and the first inequality follows from Corollary 2.5 with $q=2/\delta$ and $p=p_2/\delta$.

Now, we turn to the estimate for $H^{p_1}_{\mathcal{F},w}(\mathbb{R}^N)$ norm of b. Set $\widetilde{\Omega}_l = \{x \in \mathbb{R}^N : \mathcal{M}_{\mathcal{F}}(\chi_{\Omega_l}) > \frac{1}{2}\},\ l \in \mathbb{Z}$. Then the desired estimate follows from

$$\Big\| \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_l^{J,K}} |R|\phi_J * h(x_R)\phi_J(\cdot - x_R)\Big\|^{p_1}_{H^{p_1}_{\mathcal{F},w}(\mathbb{R}^N)} \lesssim (2^l\alpha)^{p_1} w(\widetilde{\Omega}_{l-1}), \tag{4.2}$$

for any $0 < p_1 \le 1$ and $l \ge 1$. Indeed, by the $L^q_w(\mathbb{R}^N)$, $q > q_w$, boundedness of $\mathcal{M}_{\mathcal{F}}$,

$$w(\widetilde{\Omega}_{l-1}) \lesssim \int_{\mathbb{R}^N} [\mathcal{M}_{\mathcal{F}}(\chi_{\Omega_{l-1}})(x)]^q w(x)dx \lesssim w(\Omega_{l-1}).$$

This fact together with (4.2) yields

$$\begin{aligned}\|b\|^{p_1}_{H^{p_1}_{\mathcal{F},w}(\mathbb{R}^N)} &\lesssim \sum_{l\ge1}(2^l\alpha)^{p_1} w(\widetilde{\Omega}_{l-1}) \lesssim \sum_{l\ge1}(2^l\alpha)^{p_1} w(\Omega_{l-1}) \lesssim \int_{\{\widetilde{g}_{\mathcal{F}}(f)(x)>\alpha\}} [\widetilde{g}_{\mathcal{F}}(f)(x)]^{p_1} w(x)dx \\ &\lesssim \alpha^{p_1-p} \int_{\{\widetilde{g}_{\mathcal{F}}(f)(x)>\alpha\}} [\widetilde{g}_{\mathcal{F}}(f)(x)]^{p} w(x)dx \lesssim \alpha^{p_1-p}\|f\|^p_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}.\end{aligned}$$

Thus to finish the proof, it remains to establish (4.2). Following the same argument as in the estimation of

$$\Big\|\Big\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} |(\psi_J * g)(x_R)|^2\chi_R\Big\}^{\frac{1}{2}}\Big\|_{L^{p_2}_w(\mathbb{R}^N)},$$

we get

$$\Big\| \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_l^{J,K}} |R|\phi_J * h(x_R)\phi_J(\cdot - x_R)\Big\|_{H^{p_1}_{\mathcal{F},w}} \lesssim \Big\|\Big\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_l^{J,K}} |\phi_J * h(x_R)|^2\chi_R\Big\}^{\frac{1}{2}}\Big\|_{L^{p_1}_w}. \tag{4.3}$$

Note that $R \subset \widetilde{\Omega}_{l-1}$ for $R \in \mathcal{R}_l^{J,K}$. Thus, $|R \cap (\widetilde{\Omega}_{l-1} \setminus \Omega_l)| > \frac{1}{2}|R|$, which implies

$$\chi_R(x) \le 2^{\frac{1}{\delta}} \mathcal{M}_{\mathcal{F}}(\chi_{R\cap(\widetilde{\Omega}_{l-1}\setminus\Omega_l)})^{\frac{1}{\delta}}(x).$$

As in the proof of claim (4.1), choosing $\delta < \min\{2, p_1/q_w\}$ and applying Corollary 2.5, we have

$$\begin{aligned}(2^l\alpha)^{p_1} w(\widetilde{\Omega}_{l-1}) &\ge \int_{\mathbb{R}^N} \Big\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} |\phi_J * h(x_R)|^2 \chi_{R\cap(\widetilde{\Omega}_{l-1}\setminus\Omega_l)}(x)\Big\}^{\frac{p_1}{2}} w(x)dx \\ &\gtrsim \int_{\mathbb{R}^N} \Big\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_l^{J,K}} |\phi_J * h(x_R)|^2 \mathcal{M}_{\mathcal{F}}(\chi_{R\cap(\widetilde{\Omega}_{l-1}\setminus\Omega_l)})^{\frac{2}{\delta}}(x)\Big\}^{\frac{p_1}{2}} w(x)dx \\ &\gtrsim \Big\|\Big\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_l^{J,K}} |\phi_J * h(x_R)|^2\chi_R\Big\}^{\frac{1}{2}}\Big\|^{p_1}_{L^{p_1}_w(\mathbb{R}^N)}.\end{aligned}$$

Combining this with (4.3) yields (4.2), and hence Theorem 1.7 follows. □

We end this paper with the

Proof of Theorem 1.8. Suppose that T is bounded from $H^{p_1}_{\mathcal{F},w}(\mathbb{R}^N)$ to $L^{p_1}_w(\mathbb{R}^N)$ and bounded from $H^{p_2}_{\mathcal{F},w}(\mathbb{R}^N)$ to $L^{p_2}_w(\mathbb{R}^N)$. Given $f \in H^p_{\mathcal{F},w}(\mathbb{R}^N), p_1 < p < p_2$, the Calderón-Zygmund decomposition shows that $f = g + b$ with $\|g\|^{p_2}_{H^{p_2}_{\mathcal{F},w}(\mathbb{R}^N)} \lesssim \alpha^{p_2-p}\|f\|^p_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}$ and $\|b\|^{p_1}_{H^{p_1}_{\mathcal{F},w}(\mathbb{R}^N)} \lesssim \alpha^{p_1-p}\|f\|^p_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}$. Moreover, in the proof of Theorem 1.7, we have obtained

$$\|g\|^{p_2}_{H^{p_2}_{\mathcal{F},w}(\mathbb{R}^N)} \lesssim \int_{\{\widetilde{g}_{\mathcal{F}}(f)(x)\le\alpha\}} [\widetilde{g}_{\mathcal{F}}(f)(x)]^{p_2} w(x)dx$$

and

$$\|b\|^{p_1}_{H^{p_1}_{\mathcal{F},w}(\mathbb{R}^N)} \lesssim \int_{\{\widetilde{g}_{\mathcal{F}}(f)(x)>\alpha\}} [\widetilde{g}_{\mathcal{F}}(f)(x)]^{p_1} w(x)dx.$$

Therefore,

$$\begin{aligned}
\|Tf\|^p_{L^p_w(\mathbb{R}^N)} &\le p\int_0^\infty \alpha^{p-1} w(\{x : |T(g)(x)| > \frac{\alpha}{2}\})d\alpha \\
&\quad + p\int_0^\infty \alpha^{p-1} w(\{x : |T(b)(x)| > \frac{\alpha}{2}\})d\alpha \\
&\lesssim \int_0^\infty \alpha^{p-1}\Big(\frac{\|T(g)\|_{L^{p_2}_w}}{\alpha}\Big)^{p_2} d\alpha + \int_0^\infty \alpha^{p-1}\Big(\frac{\|T(b)\|_{L^{p_1}_w}}{\alpha}\Big)^{p_1} d\alpha \\
&\lesssim \int_0^\infty \alpha^{p-p_2-1} \int_{\{\widetilde{g}_{\mathcal{F}}(f)(x)\le\alpha\}} [\widetilde{g}_{\mathcal{F}}(f)(x)]^{p_2} w(x)dxd\alpha \\
&\quad + \int_0^\infty \alpha^{p-p_1-1} \int_{\{\widetilde{g}_{\mathcal{F}}(f)(x)>\alpha\}} [\widetilde{g}_{\mathcal{F}}(f)(x)]^{p_1} w(x)dxd\alpha \\
&\lesssim \|f\|^p_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}.
\end{aligned}$$

Thus, $\|Tf\|_{L^p_w(\mathbb{R}^N)} \lesssim \|f\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}$ for any $p \in (p_1, p_2)$. Hence T is bounded from $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ to $L^p_w(\mathbb{R}^N)$.

To prove the second assertion that T is bounded on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ for $p \in (p_1, p_2)$, for any given $\alpha > 0$ and $f \in H^p_{\mathcal{F},w}(\mathbb{R}^N)$, we apply the Calderón-Zygmund decomposition again to obtain

$$\begin{aligned}
w(\{x : |\widetilde{g}_{\mathcal{F}}(Tf)(x)| > \alpha\}) &\le w(\{x : |\widetilde{g}_{\mathcal{F}}(Tg)(x)| > \alpha/2\}) + w(\{x : |\widetilde{g}_{\mathcal{F}}(Tb)(x)| > \alpha/2\}) \\
&\lesssim \alpha^{-p_2}\|T(g)\|^{p_2}_{H^{p_2}_{\mathcal{F},w}(\mathbb{R}^N)} + \alpha^{-p_1}\|T(b)\|^{p_1}_{H^{p_1}_{\mathcal{F},w}(\mathbb{R}^N)} \\
&\lesssim \alpha^{-p_2}\|g\|^{p_2}_{H^{p_2}_{\mathcal{F},w}(\mathbb{R}^N)} + \alpha^{-p_1}\|b\|^{p_1}_{H^{p_1}_{\mathcal{F},w}(\mathbb{R}^N)} \\
&\le \alpha^{-p_2}\int_{\{\widetilde{g}_{\mathcal{F}}(f)(x)\le\alpha\}} [\widetilde{g}_{\mathcal{F}}(f)(x)]^{p_2} w(x)dx \\
&\quad + \alpha^{-p_1}\int_{\{\widetilde{g}_{\mathcal{F}}(f)(x)>\alpha\}} [\widetilde{g}_{\mathcal{F}}(f)(x)]^{p_1} w(x)dx,
\end{aligned}$$

which, as above, shows that $\|\widetilde{g}_{\mathcal{F}}(Tf)\|_{L^p_w(\mathbb{R}^N)} \lesssim \|f\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}$ and hence, $\|Tf\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)} \lesssim \|f\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}$ for any $p \in (p_1, p_2)$. The proof of Theorem 1.8 is complete. $\square$

5. Appendix: Relations among different classes of weights

In this appendix, we clarify the relations between different classes of weights by constructing some examples/counterexamples.

Proposition 5.1. *For $1 < p < \infty$,*

$$A_p^{\mathrm{pro}}(\mathbb{R}^N) \subsetneq A_p^{\mathcal{F}}(\mathbb{R}^N) \subsetneq A_p(\mathbb{R}^N).$$

To prove this proposition, we need the following lemma.

Lemma 5.2. (i) *If $-n < a$ and $A > 0$, then $(|x| + A)^a dx$ is a doubling measure with doubling constant depending on the doubling constant of $|x|^a dx$ and n, but uniformly in A;*

(ii) *If $-n < a < n(p-1)$ and $A > 0$, then $(|x| + A)^a$ is an A_p weight uniformly in A.*

Proof. If $A = 0$, the above claims are well known. Let us prove (i) first. For any $A > 0$, fix A and divide all balls $B(x_0, R)$ in $\mathbb{R}^n$ into two categories: balls of type I that satisfy $|x_0| + A \geq 3R$ and type II that satisfy $|x_0| + A < 3R$.

For the balls of the first type, we have

$$\int_{B(x_0,2R)} (|x| + A)^a dx \lesssim \begin{cases} R^n(|x_0| + A + 2R)^a, & \text{if } a \geq 0 \\ R^n(|x_0| + A - 2R)^a, & \text{if } a < 0 \end{cases} \tag{5.1}$$
$$\lesssim R^{n+a}$$

and

$$\int_{B(x_0,R)} (|x| + A)^a dx \gtrsim \begin{cases} R^n(|x_0| + A - 2R)^a, & \text{if } a \geq 0 \\ R^n(|x_0| + A + 2R)^a, & \text{if } a < 0 \end{cases} \tag{5.2}$$
$$\gtrsim R^{n+a},$$

from which the doubling property follows.

For balls of the second type, we have $|x_0| + A \leq 3R$. Therefore

$$\int_{B(x_0,2R)} (|x| + A)^a dx \leq \int_{B(0,5R)} (|x| + A)^a dx$$
$$\approx \int_0^{5R} (r + A)^a r^{n-1} dr \lesssim R^{n+a}$$

and

$$\int_{B(x_0,R)} (|x| + A)^a dx \gtrsim \begin{cases} \displaystyle\int_{B(0,R)} |x|^a dx, & \text{if } a \geq 0 \\ \displaystyle\int_{B(3R\frac{x_0}{|x_0|},R)} |x|^a dx, & \text{if } a < 0 \end{cases} \tag{5.3}$$
$$\gtrsim R^{n+a},$$

Hence (i) has been proved.

For (ii), we need to prove that if $-n < a < n(p-1)$, then

$$(5.4)\qquad \left(\frac{1}{|B|}\int_B (|x|+A)^a dx\right)\left(\frac{1}{|B|}\int_B (|x|+A)^{-a\frac{p'}{p}} dx\right)^{\frac{p}{p'}} < C < \infty,$$

where C is independent of A and B. To this end, we split the balls in $\mathbb{R}^n$ into balls of type I and type II as above. If $B = B(x_0, R)$ is a ball of type I, then

$$\text{LHS of (5.4)} \approx (|x_0|+A)^a \left[(|x_0|+A)^{-a\frac{p'}{p}}\right]^{\frac{p}{p'}} = 1.$$

If $B = B(x_0, R)$ is a ball of type II, then by the doubling property of $(|x|+A)^a$, we obtain

$$\begin{aligned}
\text{LHS of (5.4)} &\approx \left(\frac{1}{R^n}\int_{B(0,5R)} (|x|+A)^a dx\right)\left(\frac{1}{R^n}\int_{B(0,5R)} (|x|+A)^{-a\frac{p'}{p}} dx\right)^{\frac{p}{p'}} \\
&\approx \left(\frac{1}{R^n}\int_0^{5R} (r+A)^a r^{n-1} dr\right)\left(\frac{1}{R^n}\int_0^{5R} (r+A)^{-a\frac{p'}{p}} r^{n-1} dr\right)^{\frac{p}{p'}} \\
&\approx R^a (R^{-a\frac{p'}{p}})^{\frac{p}{p'}} = 1.
\end{aligned}$$

This concludes the proof of Lemma 5.2. □

Proof of Proposition 5.1. By definition, it is clear that $A_p^{\text{pro}}(\mathbb{R}^N) \subset A_p^{\mathcal{F}}(\mathbb{R}^N) \subset A_p(\mathbb{R}^N)$.

Now, let us show that these inclusions are proper. For simplicity, we only consider the bi-parameter case $N = n_1 + n_2$. Choose $a \in (-n_1, n_1(p-1)), b \in (0, n_2(p-1))$ such that $a + b \geq n_1(p-1)$. Let

$$w(x,y) = |x|^a (|x|+|y|)^b.$$

We claim that

$$(5.5)\qquad \operatorname*{ess\,sup}_{y\in\mathbb{R}^{n_2}} [w(\cdot,y)]_{A_p(\mathbb{R}^{n_1})} = \infty$$

$$(5.6)\qquad \sup_{x\in\mathbb{R}^{n_1}} [w(x,\cdot)]_{A_p(\mathbb{R}^{n_2})} < \infty$$

$$(5.7)\qquad [w]_{A_p(\mathbb{R}^N)} < \infty.$$

Assume that (5.5), (5.6) and (5.7) hold for the moment. Then (5.6) and (5.7) imply $w \in A_p^{\mathcal{F}_1}$ while (5.5) implies $w \notin A_p^{\text{pro}}$, and hence $A_p^{\text{pro}}(\mathbb{R}^N) \subsetneq A_p^{\mathcal{F}_1}(\mathbb{R}^N)$.

Now let us prove (5.5), (5.6) and (5.7). To verify (5.5), we note that for any $x \in \mathbb{R}^{n_1}$ and any $y \in \mathbb{R}^{n_2}$,

$$w(x,y)^{-\frac{p'}{p}} = |x|^{-\frac{ap'}{p}} (|x|+|y|)^{-\frac{bp'}{p}} \geq |x|^{-\frac{(a+b)p'}{p}}.$$

Note also that $|x|^{-\frac{(a+b)p'}{p}}$ is not integrable over $Q(0,1)$ since $-\frac{(a+b)p'}{p} < -n_1$ by our choice of a and b. Hence for any $y \in \mathbb{R}^{n_2}$,

$$\int_{Q(0,1)} w(x,y)^{-1/(p-1)} dx \geq \int_{Q(0,1)} |x|^{-(a+b)\frac{p'}{p}} dx = \infty,$$

which gives (5.5). (5.6) follows immediately from part (ii) of Lemma 5.2 whereas (5.7) follows from (5.6).

Finally, we take $c \in [n_2(p-1), N(p-1))$. It is well known that $(|x|+|y|)^c$ is in $A_p(\mathbb{R}^N)$. But $(|x|+|y|)^c \notin A_p^{\mathcal{F}_1}(\mathbb{R}^N)$ since $\operatorname*{ess\,sup}_{x\in\mathbb{R}^{n_1}} [(|x|+|\cdot|)^c]_{A_p(\mathbb{R}^{n_2})} = \infty$ by part (ii) of Lemma 5.3. This completes the proof of Proposition 5.1. $\square$

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